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The “Quantum-Butterfly Effect" from a Kinetic Equation Approach

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Nothing is too wonderful to be true, if it be consistent with the laws of nature.

Michael Faraday

Abstract

We here study chaos in many-body quantum systems. We follow a recently developed approach which builds on kinetic equations for generalized distributions. These distributions store information on so-called out-of time ordered correlation functions, which can show an exponential increase in time, the quantum butterfly effect. The latter is currently widely used as a diagnostic tool for many-body chaos, and within the kinetic approach the quantum butterfly effect manifests in an instability related to positive eigenvalues of the relevant collision integrals. The latter determine the Lyapunov rates on which many-body chaos establishes. We here derive the generalized kinetic equations using an augmented Keldysh technique. We then apply the method to study many-body chaos in a weakly interacting electron gas, and derive the Lyapunov exponent for this system.

Keywords: quantum butterfly effect, many-body chaos, kinetic equations.

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Chapter 1

Introduction

In this thesis we study signatures of chaos in many-body quantum systems. Understanding the conditions under which generic closed quantum systems reach, at some point or not, thermal equilibrium touches on a fundamental question of theoretical and practical interest. On the characterization of ergodicity of a system.

From the theory perspective thermal equilibrium is intertwined with the notion of ergodicity generated by chaotic dynamics. The possibility that closed interacting quantum system may, even under quite general conditions, be in a phase that lacks ergodicity is a topic of ongoing current debate[**?**]. In non-ergodic, "many-body localized" phases a system does not reach a thermal equilibrium and concepts of statistical physics do not apply [1]. Such "quantum glassy" phases raise fundamental questions e.g. concerning universality classes and intermediate "extended non-ergodic" states.

From a practical perspective, the impressive progress in controlling and manipulating cold atom systems has developed to a stage realizing the old idea of "quantum simulators" already envisioned by Feynman more than three decades ago [2]. That is, by now we can realize special purpose devices designed to provide insight into specific problems like the question of under which conditions the dynamics of quantum systems become ergodic [3, 4, 5].

A question one is confronted with when aiming to investigate mechanisms that can suppress ergodicity, is which tools to use to detect signatures of quantum chaos. The pioneering work by Wigner and Dyson[6], who realized that the excitation spectra of heavy nuclei have the same spacing distribution as the eigenvalues of certain random matrix ensembles, has strongly influenced the research on quantum chaos for decades. It was seen that these random matrix ensembles also share common characteristics, such as level repulsions, with classically chaotic systems. Indeed, the search for a 'proof' of the Bohigas-Giannoni-Schmidt conjecture [7] that quantum energy levels of systems with chaotic classical counterparts share the same statistical properties as random matrices dominated research during the 90's and 00's[9]. By now, comparing spectral-statistics of a many-body quantum system to that of a random matrix ensemble is a commonly used 'acid-test' to detect whether it's quantum dynamics is chaotic and ergodic. These studies rely, however, on numerical diagonalization of the many-body Hamiltonian. Due to the exponential growth of Fock space with particle number one is restricted to relatively small system sizes which may mask some of the interesting properties. Having alternative diagnostic tools is therefore of interest.

Classically, ergodicity and hyperbolocity are two important concepts characterizing chaotic domains in phase-space. (In principle, one also has mixing, which is a stronger requirement than ergodicity, because it states that for larger times we can neglect classical correlations between the initial states of a phase space point and its time evolved partner.) In ergodic Hamiltonian systems the majority of phase-space trajectories fill the shell of constant energy uniformly, and time averages of observables then coincide with energy-shell averages. Hyperbolicity, as the name suggests, is about the strong dependence of the system on its initial conditions after time evolution is made. That is, two initial states slightly shifted within the same energy shell diverge exponentially fast in time (as long as they have a non-vanishing unstable component) on a scale defined by the so-called Lyapunov exponent. This effect is popularly known as the "butterfly effect". Notice that in principle Lyapunov rates is a local property, it is the exponential separation between trajectories of two initial similar states in phase-space. However, with the requirement of ergodicity one expects them to coincide with the system, characterizing thus globally chaos for the entire system. For the reason of its definition, hyperbolicity is not properly understood in quantum mechanics, since it's defined by fixed trajectories.

Recently, a quantum version of the butterfly effect, dubbed the quantum butterfly effect has moved into the focus of research on many-body chaos. The quantum butterfly effect is observed in so called out-of-time ordered correlation functions (OTOCs). In many-body systems with Wigner Dyson spectral statistics the latter show an exponential instability. This exponential growth is characterized by a rate, which in the semiclassical limit (where defined) corresponds to the classical Lyapunov rate. This has made OTOCs and the quantum butterfly effect an interesting tool to diagnose many-body chaos, and interesting results have already been obtained, see e.g. [8], [9] and [10].

While the above notions will be explained in more detail in the next chapter we can already outline the direction followed in this thesis. We here explore a recent work by Aleiner-Faoro-Ioffe [11] where the quantum butterfly effect was studied within a kinetic equation approach. Instead of studying one specific correlation function, or more specifically one OTOC, the authors derived kinetic equations for a generalized distribution which stores the information encoded in the latter. Interestingly, the kinetic equations encoding the dynamics of the out-of-time ordered correlation functions reveal an instability. That is, in contrast to the usual kinetic equations where perturbations of equilibrium solutions relax back to the equilibrium (as guaranteed by Boltzman's H-theorem) now an exponential increase in perturbations to equilibrium solutions is encountered. This is the manifestation of the quantum butterfly effect within the kinetic equation approach! This makes the kinetic equations an interesting starting point to study conditions under which many-body chaos, viz. the instability, is suppressed.

In this thesis we first derive the kinetic equations following a slightly different route than Aleiner-Faoro-Ioffe, using path integrals and fill in gaps not explicitly discussed in the original work. We use the Keldysh approach, since it is a proper way to study the correlators and also because Keldysh allow us to study the system out of equilibrium. As a first application we then concentrate on the weakly interacting electron gas. We derive the linearized kinetic equations, and from diagonalization of the relevant collision integral identify the Lyapunov rate at which the instability manifests, i.e. quantum many-body chaos develops in this system. The present work prepares the grounds for future studies, where we intend to study within the same approach conditions under which many-body chaos may be suppressed. That is, on the one hand we will include disorder into the problem, and on the other hand study one-dimensional systems. In both cases one expects that emerging conservation laws will at some point suppress chaos and destroy ergodicity.

The thesis is structured as follows:

In Chapter 2 we introduce the idea of the out-of-time ordered correlation functions (OTOCs). These are our main motivation as they diagnose chaos. However, as we already mentioned we will not calculate specific OTOCs, but rather observe the quantum butterfly effect in an instability of the generalized kinetic equations. To explain the formalism, which allows to derive the generalized kinetic equations, we first review the Keldysh path-integral construction of generating functions for observables and illustrate the construction at the example of the harmonic oscillator.

In Chapter 3 we follow Aleiner-Faoro-Ioffe's work [11] and generalize the Keldysh construction to allow for the study of OTOCs within the path integral approach. We will concentrate again on the harmonic oscillator and encounter new types of Green's functions that will play a major role in the characterization of chaotic behaviour. We will then go forward and include interactions into the model. To get some familiarity, we will first consider the two specific examples of a ϕ^3 and ϕ^4-interaction, and derive kinetic equations using Wigner transformations. These kinetic equations are rather generic and save as a staring point for the study of an particular example in the following chapter.

In Chapter 4 we focus on the first application of the formalism. We development the perturbative kinetic equations for a weakly interacting electron gas, concentrating on contributions from the density-density and exchange channels. We first derive the standard kinetic equations and then their generalized version which encode the quantum butterfly effect. Calculating the relevant self-energies we derive collision integrals encoding the information on relaxation processes of the system for both types of equations. We then linearize the standard kinetic equation. The latter allows for an exact diagonalization and from the eigenvalues of the linearized collision integrals one can recover relaxation rates of the weakly interacting electron gas. With this in mind we then study the linearized generalized kinetic equations encoding the quantum butterfly effect. Mapping the corresponding integral equations onto a Schödinger equation with a hyperbolic potential, we derive the eigenvalues of the relevant collision intergral. The instability in the generalized kinetic equation reflects in positive eigenvalues which define the Lyapunov rates which characterize many-body chaos in the weakly interacting electron gas.

We conclude with a summary and discussion where we outline future research directions. A brief discussion on the origin and effects of the instabilities in the system is added in an Appendix. We there compare differences between the usual and augmented Keldysh constructions, to gain some preliminary insight into where information of the system "leaks" into.

Chapter 2

OTOCs, Feynman Path Integrals and Keldysh Technique

2.1 Out-of-time ordered correlation functions

Out-of-time ordered correlation functions (OTOCs), first introduced by Larkin and Ovchinnikov in the field of superconductivity [14], have recently atracted a lot of atention in different areas of physics [9][10]. Typically, the investigation of chaos in quantum mechanical systems is made in the context of the Bohigas-Giannoni-Schmit conjecture [7]. This conjecture says that we can analyse chaos studying energy level fluctuations and comparing those to that of random matrices. Recently, OTOCs have been introduced as an alternative view of chaos and been applied to interacting many-body systems [11]. OTOCs have also received a lot of recent attention e.g. in the study of quantum chaos in black holes [10] and the study of information scrambling in disordered systems [8]. In all the above examples an exponential dependence of the OTOCs in time was observed.

One example of an OTOC is the absolute square of the commutator of two local operators,

$$C(t) = \frac{1}{2}\langle [W(t), V(0)]^\dagger [W(t), V(0)] \rangle \tag{2.1}$$

$$= \frac{1}{2}\langle V^\dagger(0)W^\dagger(t)W(t)V(0) + W^\dagger(t)V^\dagger(0)V(0)W(t) - W^\dagger(t)V^\dagger(0)W(t)V(0) - V^\dagger(0)W^\dagger(t)V(0)W(t)\rangle, \tag{2.2}$$

where $\langle \mathcal{O} \rangle = Tr(e^{-\beta H}\mathcal{O})/Tr(e^{-\beta H})$ is a thermal average and $W(t) = e^{iHt}W(0)e^{-iHt}$. The second line in equation (2.2) presents an unusual correlation function, since it involves non-usual ordering in time of the operators. That is, it is “out-of-time ordered". Indeed, one notices that the operators are arranged along the time line as follows: Starting at $t = 0$ one goes forward in time to $t > 0$, then backward again to $t = 0$ and finally back to $t > 0$.

That this kind of correlation function knows about hyperbolicity in chaotic systems can be seen by specifying to a non-interacting system with classically chaotic dynamics and choosing the single-particle operators $V = \hat{p}$ and $W(t) = \hat{q}(t)$, the generalized momenta and position coordinate of the system. Going to a semi-classical limit we can approximate the commutator $[\hat{q}(t), \hat{p}]$ by the Poisson bracket and and notice that $\{q(t), p\} = \frac{\partial q(t)}{\partial q(0)}$. The classical chaotic dynamics then reflects in the "butterfly effect", i.e. exponential sensitivity to initial conditions i.e. $\frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t}$, where λ_L is the Lyapunov exponent, which gives the rate on which the chaotic behaviour becomes relevant, since the commutators squared forms the OTOC in (2.2). In more general set-ups the commutator $[W(t), V(0)]$ provides a diagnose of the growth of $W(t)$ given the effects of perturbations of $V(0)$, and the absolute square in $C(t)$ is useful to prevent undesired random phase cancellations. Notice that the OTOC is different from others diagnose tools that analyse the backward and forward evolution with slightly different hamiltonians, as e.g. the Loschmidt echo. While time ordered correlators can be observed in the laboratory, OTOCs pose a challenge to experimentation. Usually OTOCs are therefore a more theoretical concept and have also been dubbed as "computationals" rather than "observables". There has been, however, some interesting ideas to measure OTOCs which allow to mimic the inversion of time direction [15] and [16].

Following Aleiner-Faoro-Ioffe, the unusual time ordering will in this thesis be handled in a elegant manner generalizing what is known as the Keldysh formalism [?]. This extension of the common Keldysh formalism naturally accounts for the unusual time-ordering by the construction of the formalism. We will explain this in the next chapter. Before getting there, we first review the conventional Keldysh approach in the next section.

2.2 Keldysh field theory: Construction of the formalism for an harmonic oscillator

To review the Keldysh formalism, we here discuss the construction of a Keldysh generating function for observables for one of the most simple many-particle systems, that is, the bosonic harmonic oscillator. This section serves us to introduce all the technical steps to be implemented in the then following chapter where we introduce the augmented Keldysh theory which allows to address OTOCs.

The Keldysh formalism originated to answer the question about the "leak" of information intrinsic to quantum systems. It is generally stated that the Schödinger equation is only valid for closed systems. In practice there is usually some weak interactions or couplings to other degrees of freedom which are neglected in the study of a system. In this case the system is not completely closed and information will leak out of the system and introduce irreversibility into the system. Observables in this case can be often calculated starting out from some distribution function and the kinetic equations they follow. Within the Keldysh approach the kinetic equations can be derived naturally from a microscopic theory as we then review in detail in the next chapter.

2.2.1 Keldysh time-contour

The central idea of the Keldysh approach is to study time evolution along a time contour that goes from a time in $-\infty$, evolves to a time $t > 0$ and then returns from t to $-\infty$, see Figure (2.1). Here one assumes that the system was initially prepared, at $t = -\infty$, in a non-interacting known state. Then, after some time the interactions were slowly switched on, and the density matrix $\hat{\rho}$ of the system evolves according to the von Neumann equation

$$\partial_t \hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)]$$

where $\hat{H(t)}$ is the hamiltonian of the system. The above equation is formally solved by $\hat{\rho} = \hat{U}_{t,-\infty}\hat{\rho}_{-\infty}\hat{U}^{\dagger}_{t,-\infty} = \hat{U}_{t,-\infty}\hat{\rho}_{-\infty}\hat{U}_{-\infty,t}$ and the basic idea is to view this evolution along the Keldysh contour consisting of two lines connected at $t = \infty$, and by a region representing the density operator at $t = -\infty$ and shown in Fig. 2.1.

Why it is natural to consider time evolution along the Keldysh contour can seen by looking

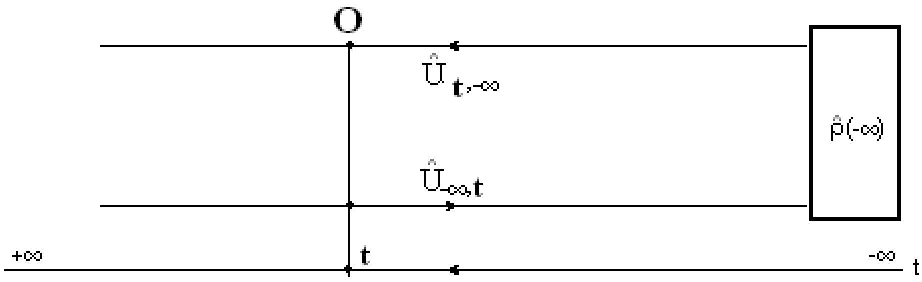

Figure 2.1: A time line describing evolution forward in time goes from $t = -\infty$ to $t = \infty$, and then winds backs on a time line describing evolution backward in time from $t = \infty$ to $t = -\infty$. The region connecting lines at at $t = -\infty$ represents the density matrix of the initial equilibrium state. The connection at $t = \infty$ is not shown here, see also Fig. 2.2.

how observables evolve in time,

$$\langle \hat{\mathcal{O}} \rangle (t) \equiv \frac{Tr(\hat{\mathcal{O}}\hat{\rho}(t))}{Tr(\hat{\rho}(t))} = \frac{1}{Tr(\hat{\rho}(t))} \; Tr(\hat{\mathcal{O}}\hat{U}_{t,-\infty}\hat{\rho}(-\infty)\hat{U}_{-\infty,t}) \tag{2.3}$$

Using cyclic invariance of the trace

$$\langle \hat{\mathcal{O}} \rangle (t) = \frac{1}{Tr(\hat{\rho}(-\infty))} \; Tr(\hat{U}_{-\infty,t}\hat{\mathcal{O}}\hat{U}_{t,-\infty}\hat{\rho}(-\infty)) \tag{2.4}$$

, with $\hat{U}_{t,-\infty} = \hat{U}^{\dagger}_{-\infty,t}$and the expression under the last trace describes (read from right to left) evolution from $t = -\infty$, where the initial density matrix is specified, toward t, where the observable is calculated, and then back to $t = -\infty$. Therefore, calculations of observables implies evolving the initial state both forward and backward. This is the closed time contour depicted in Figure (2.2). Notice that such forward-backward evolution is avoided in the usual equilibrium quantum field theory with a special trick or the use of fluctuation-dissipation theorems which do not apply straightforwardly under more general out-of equilibrium situations [?]. The Keldysh formalism is, therefore, a very powerful technique. To really make it work we have to see how to construct the time evolution operator. This can be done in the standard way by discretizing the time contour, as we discuss next.

2.2.2 Discretizing the time-contour

Discretizing the time line into a finite number of discrete time-steps δt the time evolution operators $\hat{U}_{t,t'}$ becomes a product of discrete time step evolution operators,

$$\hat{U}_{t,t'} = e^{-i\hat{H}(t-\delta_t)\delta_t} e^{-i\hat{H}(t-2\delta_t)\delta_t} ... e^{-i\hat{H}(t-N\delta_t)\delta_t} e^{-i\hat{H}(t')\delta_t} \tag{2.5}$$

The original evolution is recovered upon taking the limit of infinite many time steps

$$\hat{U}_{t,t'} = \lim_{N\to\infty} e^{-i\hat{H}(t-\delta_t)\delta_t} e^{-i\hat{H}(t-2\delta_t)\delta_t} ... e^{-i\hat{H}(t-N\delta_t)\delta_t} e^{-i\hat{H}(t')\delta_t} \tag{2.6}$$

$$= \mathbb{T} \exp\left(-i \int_{t'}^{t} \hat{H}(t) dt\right), \;\; \delta_t = (t-t')/N \tag{2.7}$$

Notice that for convenience we will always subdivide the Keldysh contour into an even number of $(2N-2)$ equal partitions of δ_t, see Fig. (2.2).
To make the construction more concrete we will now concentrate on the specific example of a bosonic harmonic oscillator. The second quantized hamiltonian for the harmonic oscillator is given by $\hat{H} = \omega_0 \hat{b}^\dagger \hat{b}$, where $\hat{b}^\dagger$ and $\hat{b}$ are the bosonic creation and annihilation operators fulfilling the commutator relation $[\hat{b}, \hat{b}^\dagger] = 1$. We then begin defining the equilibrium partition function associated to the Hamiltonian $\hat{H}$ as

$$Z = \frac{Tr(\hat{U}_k \hat{\rho})}{Tr(\hat{\rho_0})} = \frac{\langle\phi|\hat{U}_k \hat{\rho}|\phi\rangle}{\langle\phi|\hat{\rho}|\phi\rangle}. \tag{2.8}$$

Here we have $\hat{U}_k$ as the evolution operator along the Keldysh contour, and $|\phi\rangle$ a coherent state to be properly defined above. Having $\hat{\rho}_0 = e^{-\beta(\omega_0-\mu)}$ as the equilibrium density matrix we end up with

$$Tr(\hat{\rho}_0) = Tr(\exp[-\beta(\hat{H} - \mu\hat{N})]) = \sum_{n=0}^{\infty} e^{-\beta(\omega_0-\mu)n} = [1-\rho_0]^{-1} \tag{2.9}$$

Here we used that the bosonic operator algebra implies that $\hat{b}^\dagger\hat{b}|n\rangle = n|n\rangle$, $\hat{N} = \hat{b}^\dagger\hat{b}$ and $\beta = T^{-1}$, and summation of a geometric progression with common ratio between 0 and 1. On the other hand, we can use the properties of bosonic coherent states to write the identity operator in Fock space as

$$1 = \int\int \frac{d(Re\phi_j)d(Im\phi_j)}{\pi} \; \exp(-|\phi_j|^2)|\phi_j\rangle\langle\phi_j|. \tag{2.10}$$

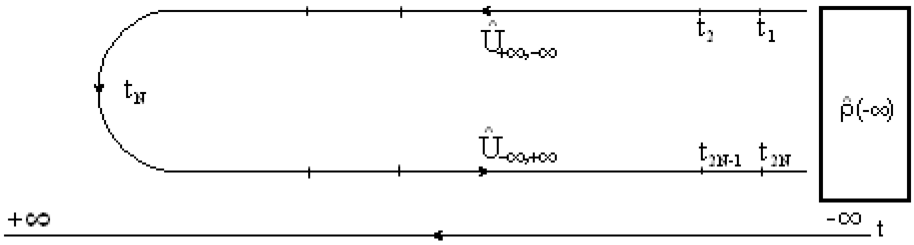

Figure 2.2: The discretized Keldysh time-contour. Here we show explicitly the connection of the two lines at $t = \infty$.

With these preparations we now study the time evolution along the Keldysh contour Figure (2.2). We start from the top following to the left, and then making the curve and getting back to the right. Going from $-\infty$ to $+\infty$, the discrete time steps are marked as t_1 to t_N, and returning to $-\infty$, marked as t_{N+1} to t_{2N}. The idea in the construction of a path-integral for the time evolution operator is to make insertions of the identity operator (2.10) at each point of the contour, and to sum the dynamic phase factors accumulated during the $(2N-2)$ time-intervals. For $N = 4$ this procedure gives e.g. the following contribution,

$$\langle\phi_8|\hat{U}_{-\delta t}|\phi_7\rangle\langle\phi_7|\hat{U}_{-\delta t}|\phi_6\rangle\langle\phi_6|\hat{U}_{-\delta t}|\phi_5\rangle\langle\phi_5|\hat{I}|\phi_4\rangle\langle\phi_4|\hat{U}_{+\delta t}|\phi_3\rangle\times$$

$$\times\,\langle\phi_3|\hat{U}_{+\delta t}|\phi_2\rangle\langle\phi_2|\hat{U}_{+\delta t}|\phi_1\rangle\langle\phi_1|\hat{\rho}_0|\phi_8\rangle. \tag{2.11}$$

The inclusion of the terms $|\phi_j\rangle\langle\phi_j|$, according to (2.10) has added to the expression above terms of the type $\exp(-|\phi_j|^2)$ that latter will be discounted to the expression. Equation (2.11) can be calculated for a given ϕ_j, following that we have for coherent states $b|\phi\rangle = \phi|\phi\rangle$ e $\langle\phi|\hat{b}^\dagger = \langle\phi|\phi$, $\langle\phi|\phi'\rangle = e^{\phi\phi'}$, since $|\phi\rangle = \sum_{n=0}^{\infty}\frac{\phi^n}{\sqrt{n!}}|n\rangle$. After a few calculations the final result will be obtained in the following procedure, for $i \neq j$:

$$\langle\phi_i|\hat{U}_{\pm\delta t}|\phi_j\rangle = \langle\phi_i|e^{\mp i\hat{H}\delta_t}|\phi_j\rangle = \langle\phi_i|(1 \mp i\hat{H}\delta_t)|\phi_j\rangle = e^{\phi_i(1\mp i\omega_0\delta_t)\phi_j} \tag{2.12}$$

Notice that there is a need to discount the exponential term originated by the structure of the bosonic coherent state. That is, the terms $|\phi_j\rangle\langle\phi_j|$ included in the partitions of the correlator have now to be discounted by use of (2.10). With these results in mind, the successive insertions of equation (2.10) together with (2.12) leads to the partition function, by construction in the following form

$$Z = \frac{1}{Tr[\hat{\rho}_0]} \int \int \prod_{i=1}^{4N} \left[\frac{d(Re\phi_i) d(Im\phi_i)}{\pi} \right] \exp \left(i \sum_{i,j=1}^{4N} \phi_i G_{ij}^{-1} \phi_j \right) \tag{2.13}$$

We here recognize the exponentiated action, iS_0, with a $2N \times 2N$ matrix G^{-1}. For $N = 4$ the latter is defined as

$$iG_{ij}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_0 \\ e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 \end{bmatrix}, \quad h = i\omega_0 \delta_t \tag{2.14}$$

Noting that, by standard determinant definitions,

$$det(iG_{ij}) = 1 - \rho(\omega_0) e^{-3h} e^{3h} = 1 - \rho(\omega_0). \tag{2.15}$$

where we neglected small contributions in δt. We readily verify that in the absence of sources, as disorder or interactions entering as contributions to the exponential in (2.13) the Keldysh partition function is always normalized

$$Z = \frac{1}{Tr(\rho_0)\ det(-iG^{-1})} \overset{(2.9),(2.15)}{=} 1 \tag{2.16}$$

Non-trivial results can be obtained by introducing sources into the action, which can then be used to generate observables. Recalling the generic rule for Gaussian integrals

$$Z(\bar{J}, J) = \int \prod_{j=1}^{N} d(\bar{z}_j, z_j) \exp(- \sum_{ij}^{N} \bar{z}_i \hat{A}_{ij} z_j + \sum_{j}^{N} (\bar{z}_j J_j + \bar{J}_j z_j)) = \frac{\exp(\sum_{ij}^{N} \bar{J} i (\hat{A}_{ij}^{-1}) J_j)}{det \hat{A}}. \tag{2.17}$$

and then noting that

$$\langle z_i \hat{z}_j \rangle \equiv \frac{1}{Z(0.0)} \frac{\delta^2 Z(\bar{J}J)}{\delta \bar{J}_i \delta J_j} \bigg|_{J=0} = \hat{A}_{ij}^{-1} \tag{2.18}$$

one can calculate the components of the Green's function G_{ij}. To this end we have to invert the matrix in the action iS_0,

$$G_{ij} = \frac{1}{1-\rho_0}\times$$

$$\times \begin{bmatrix} 1 & -\rho_0 e^{h} & \rho_0 e^{2h} & \rho_0 e^{3h} & \rho_0 & \rho_0 e^{h} & \rho_0 e^{2h} & \rho_0 e^{3h} \\ e^{-h} & 1 & \rho_0 e^{h} & \rho_0 e^{2h} & \rho_0 e^{-h} & \rho_0 & \rho_0 e^{h} & \rho_0 e^{2h} \\ e^{-2h} & e^{-h} & 1 & \rho_0 e^{h} & \rho_0 e^{-2h} & \rho_0 e^{-h} & \rho_0 & \rho_0 e^{h} \\ e^{-3h} & e^{-2h} & e^{-h} & 1 & \rho_0 e^{-3h} & \rho_0 e^{-2h} & \rho_0 e^{-h} & \rho_0 \\ 1 & e^{h} & e^{2h} & e^{3h} & 1 & \rho_0 e^{h} & \rho_0 e^{2h} & \rho_0 e^{3h} \\ e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & 1 & \rho_0 e^{h} & \rho_0 e^{2h} \\ e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & \rho_0 e^{h} \\ e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 \end{bmatrix} \tag{2.19}$$

We can then read of the propagator G_{ij},

$$\langle \phi_i^+ \phi_j^- \rangle \rho_0 = \frac{\rho_0}{1-\rho_0} \exp[-(i-j)h]$$

$$\langle \phi_j^- \phi_j^+ \rangle = \frac{\rho_0}{1-\rho_0} \exp[-(i-j)h]$$

$$\langle \phi_i^+ \phi_j^+ \rangle = \frac{1}{1-\rho_0}\left[\frac{1}{2}\delta_{ij} + \theta(i-j)\rho_0 e^{[-(i-j)h]} + \theta(j-i) e^{[-(i-j)h]}\right]$$

$$\langle \phi_i^- \phi_j^- \rangle = \frac{1}{1-\rho_0}\left[\frac{1}{2}\delta_{ij} + \theta(j-i)\rho_0 e^{[-(j-i)h]} + \theta(i-j) e^{[-(i-j)h]}\right].$$

Notice that we have here introduced subscripts $\pm$ according to a given branch of the Keldysh contour. That is, recall that the indices i, j in the propagators are the discrete time steps, which live on either the upper component of the Keldysh contour (where evolution is forward in time) or the lower component of the Keldysh contour (where evolution is backward in time). In this way there is a natural block 2×2 structure according to the component of the contour. We here use $+/-$ to refer to the contour with forward/backward evolution.

To give the above correlation function, that is, the propagators, a more physical interpretation one further step is needed. This is the so-called Keldysh rotation, to be developed in the next section. Once this has been done we can implement the continuum limit $N \to \infty$ to derive the propagators encoding time evolution on the Keldysh contour.

2.2.3 Keldysh Rotation: Retarded, advanced and Keldysh Green's functions

It is convenient to write the Green's functions in a more physical basis, obtained from the original by what is called a Keldysh rotation. Within the 2×2 block structure of the matrix G_{ij} discussed above, the Keldysh rotation is given by the following matrix

$$\hat{R} = \exp\left(\frac{i\pi\tau_2^k}{4}\right) = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \tag{2.20}$$

where $\hat{\tau}_2^k$ is the second Pauli matrix with "k" indicating the keldysh contour domain. That is, the rotation matrix acts on four subblocks blocks of the matrix, where each subblock is defined by the respective correlator introduced in the previous section

$$\hat{G}_{ij} = \begin{bmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{bmatrix}$$

Considering the fields as a vector that can be transformed by $\hat{R}$ we have

$$\begin{bmatrix} \phi_a \\ \phi_b \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix},$$

and solving this the new rotated fields read

$$\phi^{cl} = \frac{1}{\sqrt{2}}(\phi_+ + \phi_-) \tag{2.21}$$

$$\phi^{q} = \frac{1}{\sqrt{2}}(\phi_- - \phi_+). \tag{2.22}$$

$$\tag{2.23}$$

Here "cl" stands for "classical" and "q" for "quantum" components of the fields. Since the quantum one stands for the difference between the forward an backward propagations, for this has its name, because the difference between these backward and forward fields are the basis for quantum mechanics. Note that the quantum component vanishes for $\phi_- = \phi_+$. Rewriting the expressions for ϕ_+ and ϕ_- in terms of classical and quantum components we find the correlation functions to be

$$\langle \phi_i^{cl} \phi_j^{cl} \rangle = \frac{1}{2}\delta_{ij} + (2n_B + 1)e^{[-(j-i)h]} \tag{2.24}$$

$$\langle \phi_i^{cl} \phi_j^{q} \rangle = \theta(i-j)e^{[-(j-i)h]} \tag{2.25}$$

$$\langle \phi_i^{q} \phi_j^{cl} \rangle = -\theta(j-i)e^{[-(j-i)h]} \tag{2.26}$$

$$\langle \phi_i^{q} \phi_j^{q} \rangle = 0. \tag{2.27}$$

We already see here how the bosonic equilibrium occupation number n_B appears in the first correlation function. The latter is therefore of specific interest once we want to derive kinetic equations.

Similarly, the rotated Green's function matrix has the following structure, where $d = (2n_B + 1) = \frac{1+\rho}{1-\rho}$:

$$i\hat{G}' = \begin{bmatrix} d+\frac{1}{2} & de^{h} & de^{2h} & de^{3h} & \frac{1}{2} & e^{h} & e^{2h} & e^{3h} \\ de^{-h} & d+\frac{1}{2} & de^{h} & de^{2h} & 0 & \frac{1}{2} & e^{h} & e^{2h} \\ de^{-2h} & de^{-h} & d+\frac{1}{2} & de^{h} & 0 & 0 & \frac{1}{2} & e^{h} \\ de^{-3h} & de^{-2h} & de^{-h} & d+\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e^{-h} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -e^{-2h} & -e^{-h} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -e^{-3h} & -e^{-2h} & -e^{-h} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.28}$$

Reading the blocks from left to right and from top to down we identify: The first block is a full matrix with no zero elements. This is called the Keldysh Green's function, iG^K. this is the only one that stores information on the distribution function n_B and therefore plays a central role once we come to the derivation of kinetic equations. The following one has an upper triangular matrix that is the retarded Green's function, $i\hat{G}^R$. The lower triangular matrix is the advanced Green's function, $i\hat{G}^A$. The last block on $\hat{G}'$ is made of only zeros.

Finally, we implement the continuum limit $N \to \infty$. Bearing in mind that increasing N will not change the general matrix structure discussed above, we can consider $N \to \infty$ while keeping $N\delta_t$ constant (so that δ_t will be reduced), to arrive at a continuous time-evolution. To develop this we define $t_j = j\delta_t$, given that $h = i\omega_0\delta_t$. This will e.g. lead to the transformation $\exp[-(i-j)h] \rightarrow \exp[-i\omega_0(t-t')]$ and the matrix

$$\hat{G}' \begin{bmatrix} G^K & G^R \\ G^A & 0 \end{bmatrix},$$

with components

$$G^K = -i(2n_B + 1) \exp^{-i\omega_0(t-t')} \tag{2.29}$$

$$G^R = -i\theta(t-t') \exp^{-i\omega_0(t-t')} \tag{2.30}$$

$$G^A = i\theta(t'-t) \exp^{-i\omega_0(t-t')} \tag{2.31}$$

As already mentioned above the interesting observation is that the Keldysh Green's function stores the distribution function. This knows about the thermal equilibrium and one can derive it's governing equation, i.e. the kinetic equations storing information on equilibration and thermalization processes [13].

In a similar way, one can generalize the Keldysh contour to account for out of time ordered correlation functions. In this generalized version a new kind of distribution will appear which then stores the information on the quantum butterfly effect. The main interest then is to derive the corresponding governing equations which will store the information on many-body chaos in the system. In the next chapter we will follow this route, i.e. (i) generalize the time-contour, (ii) work out the generalized matrix propagator which now consists of more components involving extra distribution functions encoding information on OTOCs, and (iii) derive the kinetic equations for all the distribution functions appearing in the propagator.

Chapter 3

Augmented Keldysh Formalism

Having reviewed the basic construction of the path integral along the conventional Keldysh time-contour in the previous chapter, in this section we follow the work by Aleiner-Faoro-Ioffe [11] and generalize the contour in a way that allows to address OTOCs. The generalized contour is shown in Figure (3.1). Starting to read the contour at the first line on the top, one follows the contour and let time go from t_1 to t_2, from t_2 to t_1, then t_1 to t_2 and t_2 to t_1 again. That is, one considers a contour that allows for an extra time reversal. This is exactly taylored for the study of OTOCs and, therefore, the quantum butterfly effect. Effectively, we have now two coupled Keldysh contours, one for an "upper" world and on for a "downer" world. We expect that the results with respect to each world are the same as in the canonical Keldysh construction. The coupling of the two worlds will however introduce new distribution functions. This is similar as in the conventional Keldysh formalism where the coupling between the forward and backward contour introduced the Bose distribution function. It is these distributions which then play a central role for the analysis of many-body quantum chaos.

The strategy in this chapter is as follows: We first construct the propagators describing evolution along the augmented Keldysh contour. Since we are interested in electrons in the end, we will first do the construction for the bosonic and then also for the fermionic harmonic oscillator. We then include interactions into the theory. To get some general understanding of the modification introduced by interactions we discuss general ϕ^3 and ϕ^4 contact interactions, and derive perturbatively the corresponding self-energies. With this we have everything at hand to derive kinetic equations. In this chapter we derive kinetic equation for the bosonic system, but it can be easily checked calculations for fermions go

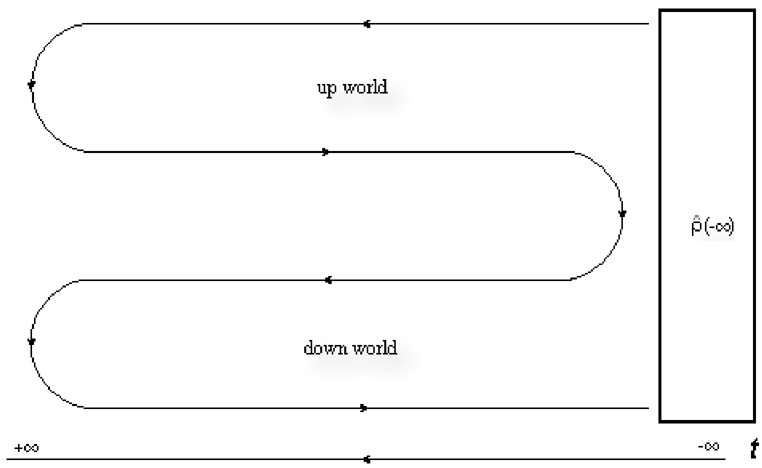

Figure 3.1: The augmented Keldysh time contour: This can be viewed as a pair of conventional Keldysh time contours, one for an “upper world" and one for an “downer world". The worlds are connected at $t = -\infty$ again by a region represented by a density operator. This latter stores information on the quantum butterfly effect.

along the same line. Since we are considering the augmented Keldysh contour, we will account three distribution functions and three corresponding kinetic equations. That is, one kinetic equation for the usual Bose, respectively Fermi, distribution also found in the conventional Keldysh formalism, and two kinetic equations for two new distributions. The latter naturally appear due to the coupling between the upper and downer worlds. From the corresponding kinetic equations we shall latter retrieve information on the quantum butterfly effect in form of an instability in the kinetic equations.

Bosons

To construct the field theory along the augmented contour we now have to account for the four time lines. Two lines describe forward propagation and two describe backward propagations in time. The four lines are connected at $t = +\infty$ (respectively $t = -\infty$), and can be viewed as two pairs of usual Keldysh contours, refered to as upper and downer worlds in the following, see also Figure (3.1). We then partition each of the time lines following the same discretization, i.e. introducing four marks at a given time, one for each line in the fourfold contour. That is, instead of $2N - 2$ partitions used in the previous chapter we here use $4N - 2$ partitions. E.g. for $N = 4$ we now get 16 marks, and the Green's function is a 16×16 matrix. Explicitly, the generalization of (2.11) now reads

$$\langle\phi_{16}|U_{-d\delta_t}|\phi_{15}\rangle\langle\phi_{15}|U_{-d\delta_t}|\phi_{14}\rangle\langle\phi_{14}|U_{-d\delta_t}|\phi_{13}\rangle\langle\phi_{13}|\hat{I}|\phi_{12}\rangle\langle\phi_{12}|U_{+d\delta_t}|\phi_{11}\rangle$$

$$\langle\phi_{11}|U_{+d\delta_t}|\phi_{10}\rangle\langle\phi_{10}|U_{+d\delta_t}|\phi_9\rangle\langle\phi_9|\hat{I}|\phi_8\rangle\langle\phi_8|U_{-u\delta_t}|\phi_7\rangle\langle\phi_7|U_{-u\delta_t}|\phi_6\rangle$$

$$\langle\phi_6|U_{-u\delta_t}|\phi_5\rangle\langle\phi_5|\hat{I}|\phi_4\rangle\langle\phi_4|U_{+u\delta_t}|\phi_3\rangle\langle\phi_3|U_{+u\delta_t}|\phi_2\rangle\langle\phi_2|U_{+u\delta_t}|\phi_1\rangle\langle\phi_1|\hat{\rho}_0|\phi_{16}\rangle. \quad (3.1)$$

The evolution operator $\hat{U}$ is of the same form as in the previous chapter. Hence, relations (2.12) are consistent with this contour since this result depends only on the hamiltonian. However, the matrix has some modifications. As one can see from the equation (3.1) there are twice as much non zero elements as that in conventional Keldysh formalism. Since ρ_0 is still the same as before, $Tr[\rho_0]$ gives the same result. With all this in mind we find the following structure for the partition function

$$Z = \frac{1}{Tr[\hat{\rho}_0]} \int\int \prod_{i=1}^{4N} \left[\frac{d(Re\phi_i^a)d(Im\phi_i^b)}{\pi^4}\right] \exp\left(i\sum_{i,j=1}^{4N} \phi_i G_{ij}^{-1}\phi_j\right) \quad (3.2)$$

where, $a, b = (clu, cld, qu, qd)$ and the $(4N \times 4N)$ G^{-1} matrix for $N = 4$ is now defined as:

$$iG_{ij}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho(\omega_{0)}) \\ e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 \end{bmatrix}, \tag{3.3}$$

where $h = i\omega_0\delta_t$. And we still have

$$Det(iG_{ij}) = 1 - \rho(\omega_{0)})e^{-6h}e^{6h} = 1 - \rho(\omega_0) \tag{3.4}$$

Following the steps of the previous chapter we then calculate the inverse of this matrix. This is simplified using (2.14) as a basis of comparison. The new Green's function matrix is then found to as

$$i\hat{G}_{ij} = \frac{1}{1-\rho} \tag{3.5}$$

$$\begin{bmatrix}
1 & -\rho e^{h} & \rho e^{2h} & \rho e^{3h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{3h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{3h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{3h} \\
e^{-h} & 1 & \rho e^{h} & \rho e^{2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{2h} \\
e^{-2h} & e^{-h} & 1 & \rho e^{h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & \rho e^{-3h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{-3h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{-3h} & \rho e^{-2h} & \rho e^{-h} & \rho \\
1 & e^{h} & e^{2h} & e^{3h} & 1 & \rho e^{h} & \rho e^{2h} & \rho e^{3h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{3h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{3h} \\
e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & 1 & \rho e^{h} & \rho e^{2h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{h} & \rho & \rho e^{-h} & \rho e^{-2h} \\
e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & \rho e^{h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & \rho e^{-3h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{-3h} & \rho e^{-2h} & \rho e^{-h} & \rho \\
1 & e^{h} & e^{2h} & e^{3h} & 1 & -e^{h} & e^{2h} & e^{3h} & 1 & \rho e^{h} & \rho e^{2h} & \rho e^{3h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{3h} \\
e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & \rho & \rho e^{h} & \rho e^{2h} & \rho e^{-h} & \rho & \rho e^{h} & \rho e^{2h} \\
e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & \rho e^{h} & \rho e^{-2h} & \rho e^{-h} & \rho & \rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & \rho e^{-3h} & \rho e^{-2h} & \rho e^{-h} & \rho \\
1 & e^{h} & e^{2h} & e^{3h} & 1 & e^{h} & e^{2h} & e^{3h} & 1 & e^{h} & e^{2h} & e^{3h} & 1 & \rho e^{h} & \rho e^{2h} & \rho e^{3h} \\
e^{-h} & 1 & e^{h} & e^{2h} & 1 & e^{-h} & e^{h} & e^{2h} & e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & 1 & \rho e^{h} & \rho e^{2h} \\
e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & -e^{-h} & 1 & \rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1
\end{bmatrix}.$$

From this matrix we can read of the correlation functions using that

$$\langle \phi_i \phi_j \rangle \equiv \int D[\phi\phi] \phi_i \phi_j \exp\left(i \sum_{i,j=1}^{4N} \phi_i G_{ij}^{-1} \phi_j \right) = iG_{ij}. \tag{3.6}$$

In this way we find

$$\begin{aligned}
\langle \phi_i^{d+} \phi_j^{u+} \rangle = \langle \phi_i^{u+} \phi_j^{d+} \rangle \rho_0 &= \frac{\rho_0}{1-\rho_0} \exp[-(i-j)h] \\
\langle \phi_i^{d-} \phi_j^{u-} \rangle = \langle \phi_i^{u-} \phi_j^{d-} \rangle \rho_0 &= \frac{\rho_0}{1-\rho_0} \exp[-(i-j)h] \\
\langle \phi_i^{u+} \phi_j^{u-} \rangle \rho_0 = \langle \phi_i^{u-} \phi_j^{d+} \rangle \rho_0 = \langle \phi_i^{d+} \phi_j^{d-} \rangle \rho_0 = \langle \phi_i^{d+} \phi_j^{u-} \rangle &= \frac{\rho_0}{1-\rho_0} \exp[-(i-j)h] \\
\langle \phi_i^{u-} \phi_j^{d+} \rangle \rho_0 = \langle \phi_i^{u-} \phi_j^{u+} \rangle = \langle \phi_i^{d-+} \phi_j^{d+} \rangle = \langle \phi_i^{d-} \phi_j^{u+} \rangle &= \frac{\rho_0}{1-\rho_0} \exp[-(i-j)h] \\
\langle \phi_i^{u+} \phi_j^{u+} \rangle = \langle \phi_i^{d+} \phi_j^{d+} \rangle &= \frac{1}{1-\rho_0} \left[\frac{1}{2} \delta_{ij} + \theta(i-j) \rho_0 e^{[-(i-j)h]} + \theta(j-i) e^{[-(i-j)h]} \right] \\
\langle \phi_i^{u-} \phi_j^{u-} \rangle = \langle \phi_i^{d-} \phi_j^{d-} \rangle &= \frac{1}{1-\rho_0} \left[\frac{1}{2} \delta_{ij} + \theta(j-i) \rho_0 e^{[-(j-i)h]} + \theta(i-j) e^{[-(i-j)h]} \right].
\end{aligned}$$

We notice the symmetries underlying the structure of the Green's function matrix. In order to make the latter explicit we apply a Keldysh rotation that now, containing the contributions for the two forward and two backward propagations, is generalized to

$$\hat{R} = \exp\left(\frac{i\pi\tau_2^k \otimes \tau_0^a}{4} \right) = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{3.7}$$

Here $\hat{\tau}_2^k$ e $\hat{\tau}_0^a$ are the following Pauli matrices, respectively:

$$\hat{\tau}_2^k = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \hat{\tau}_0^a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The rotation matrix acts on the blocks of $\hat{G}_{ij}$ for each orientation ($\pm u \pm d$) of the augmented Keldysh contours:

$$\hat{G}_{ij} = \begin{bmatrix} G_{u+u+} & G_{u+u-} & G_{u+d+} & G_{u+d-} \\ G_{u-u+} & G_{u-u-} & G_{u-d+} & G_{u-d-} \\ G_{d+u+} & G_{d+u-} & G_{d+d+} & G_{d+d-} \\ G_{d-u+} & G_{d-u-} & G_{d-d+} & G_{d-d-} \end{bmatrix}.$$

The rotation for the fields are define as:

$$\begin{bmatrix} \phi_a \\ \phi_b \\ \phi_c \\ \phi_d \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{u+} \\ \phi_{u-} \\ \phi_{d+} \\ \phi_{d-} \end{bmatrix}$$

Now we can relate the old fields to the new:

$$\phi_j^{clu} = \frac{1}{\sqrt{2}}(\phi_{u+} + \phi_{u-}) \tag{3.8}$$

$$\phi_j^{qu} = \frac{1}{\sqrt{2}}(\phi_{u-} - \phi_{u+}) \tag{3.9}$$

$$\phi_j^{cld} = \frac{1}{\sqrt{2}}(\phi_{d+} + \phi_{d-}) \tag{3.10}$$

$$\phi_j^{qd} = \frac{1}{\sqrt{2}}(\phi_{d-} - \phi_{d+}) \tag{3.11}$$

Where "cl" again refers to "classical" and "q" to "quantum" components. The new indices "u" and "d" now denote the upper and downer world, respectively. Rewriting then fields in a way that we have expressions for ϕ_+ and ϕ_- we can substitute the latter into the correlators $\langle \phi_i \phi_j \rangle$ to find new the correlators given by the classical and quantum fields.

Both of them now have upper and downer components

$$\langle \phi_i^{clu} \phi_j^{clu} \rangle = iG^K = \frac{1}{2}\delta_{ij} + (2n_B + 1)e^{[-(j-i)h]} \tag{3.12}$$

$$\langle \phi_i^{clu} \phi_j^{qu} \rangle = iG^R = \theta(i-j)e^{[-(j-i)h]} \tag{3.13}$$

$$\langle \phi_i^{qu} \phi_j^{clu} \rangle = iG^A = -\theta(j-i)e^{[-(j-i)h]} \tag{3.14}$$

$$\langle \phi_i^{cld} \phi_j^{cld} \rangle = iG^K = \frac{1}{2}\delta_{ij} + (2n_B + 1)e^{[-(j-i)h]} \tag{3.15}$$

$$\langle \phi_i^{cld} \phi_j^{qd} \rangle = iG^R = \theta(i-j)e^{[-(j-i)h]} \tag{3.16}$$

$$\langle \phi_i^{qd} \phi_j^{cld} \rangle = iG^A = -\theta(j-i)e^{[-(j-i)h]} \tag{3.17}$$

$$\langle \phi_i^{clu} \phi_j^{cld} \rangle = iG^K_{ud} = 2n_B \rho e^{[-(j-i)h]} \tag{3.18}$$

$$\langle \phi_i^{cld} \phi_j^{clu} \rangle = iG^K_{du} = 2n_B e^{[-(j-i)h]} \tag{3.19}$$

$$\tag{3.20}$$

and

$$\langle \phi_i^{qu} \phi_j^{qu} \rangle = \langle \phi_i^{qd} \phi_j^{qd} \rangle = \langle \phi_i^{clu} \phi_j^{qd} \rangle = \langle \phi_i^{qu} \phi_j^{qd} \rangle = \langle \phi_i^{qu} \phi_j^{cld} \rangle = \langle \phi_i^{cld} \phi_j^{qu} \rangle$$

$$= \langle \phi_i^{qd} \phi_j^{qu} \rangle = \langle \phi_i^{qd} \phi_j^{clu} \rangle = 0 \tag{3.21}$$

The new (rotated) matrix has then the following form (with $d = (2n_B + 1) = \frac{1+\rho}{1-\rho}$):

$$
i\hat{G}' = \tag{3.22}
$$

$$
\begin{bmatrix}
d+\frac{1}{2} & de^{h} & de^{2h} & de^{3h} & \frac{1}{2} & e^{h} & e^{2h} & e^{3h} & 2n_B\rho & 2n_B\rho e^{h} & 2n_B\rho e^{2h} & 2n_B\rho e^{3h} & 0 & 0 & 0 & 0 \\
de^{-h} & d+\frac{1}{2} & de^{h} & de^{2h} & 0 & \frac{1}{2} & e^{h} & e^{2h} & 2n_B\rho e^{-h} & 2n_B\rho & 2n_B\rho e^{h} & 2n_B\rho e^{2h} & 0 & 0 & 0 & 0 \\
de^{-2h} & de^{-h} & d+\frac{1}{2} & de^{h} & 0 & 0 & \frac{1}{2} & e^{h} & 2n_B\rho e^{-2h} & e^{-h} & 2n_B\rho & 2n_B\rho e^{h} & 0 & 0 & 0 & 0 \\
de^{-3h} & de^{-2h} & de^{-h} & d+\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 2n_B\rho e^{-3h} & 2n_B\rho e^{-2h} & 2n_B\rho e^{-h} & 2n_B\rho & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-e^{-h} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-e^{-2h} & -e^{-h} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-e^{-3h} & -e^{-2h} & -e^{-h} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2n_B & 2n_Be^{h} & 2n_Be^{2h} & 2n_Be^{3h} & 0 & 0 & 0 & 0 & d+\frac{1}{2} & de^{h} & de^{2h} & de^{3h} & \frac{1}{2} & e^{h} & e^{2h} & e^{3h} \\
2n_Be^{-h} & 2n_B\rho & 2n_Be^{h} & 2n_Be^{2h} & 0 & 0 & 0 & 0 & de^{-h} & d+\frac{1}{2} & de^{h} & de^{2h} & 0 & \frac{1}{2} & e^{h} & e^{2h} \\
2n_Be^{-2h} & e^{-h} & 2n_B & 2n_Be^{h} & 0 & 0 & 0 & 0 & de^{-2h} & de^{-h} & d+\frac{1}{2} & de^{h} & 0 & 0 & \frac{1}{2} & e^{h} \\
2n_Be^{-3h} & 2n_Be^{-2h} & 2n_Be^{-h} & 2n_B\rho & 0 & 0 & 0 & 0 & de^{-3h} & de^{-2h} & de^{-h} & d+\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2h} & e^{-h} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-3h} & e^{-2h} & e^{-h} & \frac{1}{2} & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Now it is possible to see the symmetries in the system. We have two equal diagonal blocks, which correspond to the Keldysh upper and Keldysh downer, the non-diagonal blocks which are related to the coupling of both worlds. This can be summarized in the following way:

$$i\hat{G} = \begin{bmatrix} G^K & G^R & G^K_{ud} & 0 \\ G^A & 0 & 0 & 0 \\ G^K_{du} & 0 & G^K & G^R \\ 0 & 0 & G^A & 0 \end{bmatrix}.$$

Notice that the off-diagonal blocks, connecting upper and downer worlds, only contain Keldysh Green's functions. As already noted before, the Keldysh Green's function encode information on distribution functions. It is thus the G^K_{ud} and G^K_{du} components which will be of specific relevance in the following. Before we turn to the details we first introduce the calculation for fermions.

Fermions

We can develop the same calculation for fermions. This is of interest to us because we are latter interested in problems involving electrons. To this end we use fermionic coherent states which are defined by the use of Grassmann variables. This particularity is due to the Pauli principle, which states that more than one fermion cannot be in the same quantum state. In the number basis there is then 2 orthonormal basis states, referred to as $|0\rangle$ and $|1\rangle$. As was done in the bosonic case we now introduce the fermionic version of the harmonic oscillator given by $\hat{H} = \epsilon_0 \hat{f}^\dagger \hat{f}$, where $\hat{f}^\dagger$ and $\hat{f}$ are fermionic creation and annihilation operators, respectively, following the anti-commutation relations: $\{\hat{f}, \hat{f}^\dagger\} = 1$, $\hat{f}^\dagger \hat{f}|n\rangle = n|n\rangle$ and $\hat{f}\hat{f} = \hat{f}^\dagger \hat{f}^\dagger = 0$. With these notations:

$$\hat{f}|0\rangle = 0, \qquad \hat{f}|1\rangle = |0\rangle, \qquad \hat{f}^\dagger|0\rangle = |1\rangle, \qquad \hat{f}^\dagger|1\rangle = 0. \tag{3.23}$$

Employing definitions of the Grassmann variables we have the following relations for the fermionic coherent states $\langle\psi| = \langle 0| \exp(-\hat{f}\bar{\psi})$ and $\langle\psi| = \exp(-\hat{f}^\dagger\bar{\psi})|0\rangle$ which then gives

$\langle\psi|\psi'\rangle = \exp(\psi\psi')$ and the resolution of unity:

$$1 = \int\int d\bar{\psi}_j d\psi_j \ e^{-\bar{\psi}_j\psi_j}|\psi_j\rangle\langle\psi_j|. \tag{3.24}$$

We now repeat the same construction for the propagator along the augmented Keldysh contour used for bosons, remembering that they are also non-orthogonal. That is, we use the same contour, divide it as before into $4N-2$ equal parts using discrete time steps δt, but now employing fermionic coherent states. In this way we arrive at

$$\langle\psi_{16}|U_{-d\delta_t}|\psi_{15}\rangle\langle\psi_{15}|U_{-d\delta_t}|\psi_{14}\rangle\langle\psi_{14}|U_{-d\delta_t}|\psi_{13}\rangle\langle\psi_{13}|\hat{I}|\psi_{12}\rangle\langle\psi_{12}|U_{+d\delta_t}|\psi_{11}\rangle$$

$$\langle\psi_{11}|U_{+d\delta_t}|\psi_{10}\rangle\langle\psi_{10}|U_{+d\delta_t}|\psi_9\rangle\langle\psi_9|\hat{I}|\psi_8\rangle\langle\psi_8|U_{-u\delta_t}|\psi_7\rangle\langle\psi_7|U_{-u\delta_t}|\psi_6\rangle$$

$$\langle\psi_6|U_{-u\delta_t}|\psi_5\rangle\langle\psi_5|\hat{I}|\psi_4\rangle\langle\psi_4|U_{+u\delta_t}|\psi_3\rangle\langle\psi_3|U_{+u\delta_t}|\psi_2\rangle\langle\psi_2|U_{+u\delta_t}|\psi_1\rangle\langle\psi_1|\hat{-\rho}_0|\psi_{16}\rangle. \tag{3.25}$$

Doing the same analysis as for the bosons we find

$$Z = \frac{1}{Tr[\hat{\rho}_0]}\int\int\prod_{j=1}^{4N}[d\bar{\psi}_j d\psi_j] \ \exp\left(i\sum_{i,j=1}^{4N}\bar{\psi}_i V_{ij}^{-1}\psi_j\right) \tag{3.26}$$

The trace here is given by $Tr[\hat{\rho}_0] = 1+\rho_0$ since the occupation goes only from 0 to 1, i.e. the summation only includes two terms, one for n = 0 and the other for n = 1, giving ρ_0. That is

$$Tr(\hat{\rho}_0 = \sum_{n=0}^{1}\exp[-\beta(\epsilon_0-\mu)]$$

Using the same idea as in the bosonic partition function we then write the correlation functions as a matrix,

$$Z = \int D[\bar{\psi}\psi]e^{iS[d\bar{\psi},\psi]} = \int D[\bar{\psi}\psi]\exp\left(i\int_c dt[\psi\bar{(t)}\hat{V}^{-1}\psi(t)]\right) \tag{3.27}$$

where

$$
iV_{ij}^{-1} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\rho(\epsilon_0) \\
e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-h} & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{h} & -1
\end{bmatrix} \tag{3.28}
$$

Here $h = i\epsilon_0\delta_t$, with $\rho(\epsilon_0)$ being negative here because of the anti-commutation relations. This can be seen from the last term in equation (3.25). The determinant now reads:

$$
Det(iV_{ij}) = 1 + \rho(\epsilon_{0)})e^{-6h}e^{6h} = 1 + \rho(\epsilon_0). \tag{3.29}
$$

To simplify the Keldysh rotation, to be done in the next step, we label the Green's function elements as: $(1, 2, ...N-1, N, 1, 2, ...N-1, N, 1, 2, ...N-1, N, 1, 2, ...N-1, N)$. We next calculate the inverse of the matrix and find

$$i\hat{V}_{ij} = \frac{1}{1+\rho} \tag{3.30}$$

$$\begin{bmatrix}
1 & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} \\
e^{-h} & 1 & -\rho e^{h} & -\rho e^{2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{2h} \\
e^{-2h} & e^{-h} & 1 & -\rho e^{h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho \\
1 & e^{h} & e^{2h} & e^{3h} & 1 & e^{h} & e^{2h} & e^{3h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} \\
e^{-h} & 1 & e^{h} & e^{2h} & -\rho e^{-h} & 1 & e^{h} & e^{2h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{h} & -\rho & -\rho e^{-h} & -\rho e^{-2h} \\
e^{-2h} & e^{-h} & 1 & e^{h} & -\rho e^{-2h} & -\rho e^{-h} & 1 & e^{h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & 1 & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho \\
1 & e^{h} & e^{2h} & e^{3h} & 1 & -e^{h} & e^{2h} & e^{3h} & 1 & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{3h} \\
e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & 1 & e^{h} & e^{2h} & e^{-h} & -\rho & -\rho e^{h} & -\rho e^{2h} & -\rho e^{-h} & -\rho & -\rho e^{h} & -\rho e^{2h} \\
e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & -\rho e^{h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho & -\rho e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & -\rho e^{-3h} & -\rho e^{-2h} & -\rho e^{-h} & -\rho \\
1 & e^{h} & e^{2h} & e^{3h} & 1 & e^{h} & e^{2h} & e^{3h} & 1 & e^{h} & e^{2h} & e^{3h} & 1 & e^{h} & e^{2h} & e^{3h} \\
e^{-h} & 1 & e^{h} & e^{2h} & 1 & e^{-h} & e^{h} & e^{2h} & e^{-h} & 1 & e^{h} & e^{2h} & -\rho e^{h} & 1 & -\rho e^{-h} & e^{-2h} \\
e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & e^{h} & e^{-2h} & e^{-h} & 1 & e^{h} & -\rho e^{-2h} & -\rho e^{-h} & 1 & e^{h} \\
e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & e^{-3h} & e^{-2h} & e^{-h} & 1 & -\rho e^{-3h} & -\rho e^{-2h} & e^{-h} & 1
\end{bmatrix}$$

Then we perform a Keldysh rotation, defined as

$$\hat{V}' = (\hat{R}\hat{V}\hat{R}^{\dagger}).(\hat{\tau}_1^{k} \otimes \hat{\tau}_0^{a}) \tag{3.31}$$

$$i\hat{V}' = \begin{bmatrix} V^A & V^K & 0 & V^K_{ud} \\ 0 & V^A & 0 & 0 \\ 0 & V^K_{du} & V^A & V^K \\ 0 & 0 & 0 & V^A \end{bmatrix} \tag{3.32}$$

where we have

$$V^R = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ -e^{-h} & -\frac{1}{2} & 0 & 0 \\ -e^{-2h} & -e^{-h} & -\frac{1}{2} & 0 \\ -e^{-3h} & -e^{-2h} & -e^{-h} & -\frac{1}{2} \end{pmatrix}$$

$$V^A = \begin{pmatrix} \frac{1}{2} & e^h & e^{2h} & e^{3h} \\ 0 & \frac{1}{2} & e^h & e^{2h} \\ 0 & 0 & \frac{1}{2} & e^h \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$V^K = \begin{pmatrix} \frac{3-\rho}{2(\rho+1)} & d'e^h & d'e^{2h} & d'e^{3h} \\ d'e^{-h} & \frac{3-\rho}{2(\rho+1)} & d'e^h & d'e^{2h} \\ d'e^{-2h} & d'e^{-h} & \frac{3-\rho}{2(\rho+1)} & d'e^h \\ d'e^{-3h} & d'e^{-2h} & d'e^{-h} & \frac{3-\rho}{2(\rho+1)} \end{pmatrix}$$

$$V^K_{ud} = \begin{pmatrix} 2n_F\rho - \frac{1}{2} & 2n_F\rho e^h & 2n_F\rho e^{2h} & 2n_F\rho e^{3h} \\ 2n_F\rho e^{-h} & 2n_F\rho - \frac{1}{2} & 2n_F\rho e^h & 2n_F\rho e^{2h} \\ 2n_F\rho e^{-2h} & 2n_F\rho e^{-h} & 2n_F\rho - \frac{1}{2} & 2n_F\rho e^h \\ 2n_F\rho e^{-3h} & 2n_F\rho e^{-2h} & 2n_F\rho e^{-h} & 2n_F\rho - \frac{1}{2} \end{pmatrix}$$

$$V^K_{du} = \begin{pmatrix} (\frac{5}{2} - 2n_F) & (2-2n_F)\rho e^h & (2-2n_F)\rho e^{2h} & (2-2n_F)\rho e^{3h} \\ (2-2n_F)\rho e^{-h} & (\frac{5}{2} - 2n_F) & (2-2n_F)\rho e^h & (2-2n_F)\rho e^{2h} \\ (2-2n_F)\rho e^{-2h} & (2-2n_F)\rho e^{-h} & (\frac{5}{2} - 2n_F) & (2-2n_F)\rho e^h \\ (2-2n_F)\rho e^{-3h} & (2-2n_F)\rho e^{-2h} & (2-2n_F)\rho e^{-h} & (\frac{5}{2} - 2n_F) \end{pmatrix},$$

and $d' = \frac{1-\rho}{1+\rho}$. V^A is the advanced Green's function matrix, V^R is the retarded Green's function matrix, V^K is the Keldysh Green's function matrix. They are equal and valid for the upper and downer world. The remaining matrices are the interworld matrices, connecting upper and downer worlds.

Finally, we apply the continuum limit to the above expression for bosons and fermions. Following the steps made for bosons we take N $\rightarrow \infty$ while $N\delta_t \rightarrow \textit{constant}$. For that we define $t_i = i\delta_t$ in a way that $\exp(-(j-i)h) \rightarrow \exp(-i\omega_0(t-t'))$ (bosons) or $\exp(-(j-i)h) \rightarrow \exp(-i\epsilon_0(t-t'))$ (fermions). We then have:

Fermions

$$V^K = -i(1-2n_F)\exp^{-i\epsilon_0(t-t')} \tag{3.33}$$

$$V^R = -i\theta(t-t')\exp^{-i\epsilon_0(t-t')} \tag{3.34}$$

$$V^A = i\theta(t'-t)\exp^{-i\epsilon_0(t-t')} \tag{3.35}$$

$$V^K_{du} = (2-2n_F)\exp^{-i\epsilon_0(t-t')} \tag{3.36}$$

$$V^K_{ud} = 2n_F\exp^{-i\epsilon_0(t-t')} \tag{3.37}$$

Bosons

$$G^K = -i(2n_B+1)\exp^{-i\omega_0(t-t')} \tag{3.38}$$

$$G^R = -i\theta(t-t')\exp^{-i\omega_0(t-t')} \tag{3.39}$$

$$G^A = i\theta(t'-t)\exp^{-i\omega_0(t-t')} \tag{3.40}$$

$$G^K_{du} = 2n_B\exp^{-i\omega_0(t-t')} \tag{3.41}$$

$$G^K_{ud} = 2n_B\rho\exp^{-i\omega_0(t-t')} \tag{3.42}$$

So far we have introduced the formalism for the simplest case, i.e. non-interacting bosons and fermions. Since we are interested in many-body chaos we next should include interactions.

3.1 Interactions

So let us now introduce interaction to our theory. We will consider general ϕ^4 and ϕ^3 - interactions to get familiarity with the structure of the theory. We will introduce the perturbative self-energies for both cases. The latter are important ingredients for the kinetic equations to be derived in the next section.

In the presence of interaction the partition function reads

$$Z = \frac{1}{Tr[\hat{\rho}_0]} \int D[\phi^a \phi^b] \ \exp(iS_0 + iS_{int}) \tag{3.43}$$

where S_0 and S_{int} are the free and interaction contributions to action, respectively. The free part of the theory reads

$$Z = \frac{1}{Tr[\hat{\rho}_0]} \int \prod_p \left[\frac{d(Re\phi^a(p))d(Im\phi^b(p))}{\pi^4} \right] \ \exp(iS_0) \tag{3.44}$$

with

$$S_0(\phi^{clu}, \phi^{qu}, \phi^{cld}, \phi^{qd}) = \int \int_{-\infty}^{+\infty} dtdt' (\phi^{clu} \phi^{qu} \phi^{cld} \phi^{qd})$$

$$\begin{pmatrix} 0 & G^{A^{-1}} & 0 & 0 \\ G^{R^{-1}} & G^{K^{-1}} & 0 & G^{udK^{-1}} \\ 0 & 0 & 0 & G^{A^{-1}} \\ 0 & G^{duK^{-1}} & G^{R^{-1}} & G^{K^{-1}} \end{pmatrix} \begin{pmatrix} \phi^{clu} \\ \phi^{qu} \\ \phi^{cld} \\ \phi^{qd} \end{pmatrix}. \tag{3.45}$$

and $S(\phi^{clu}, 0, \phi^{cld}, 0) = 0$ to preserve causality. We can think of the free part as just the harmonic oscillator terms discussed in the previous sections.

ϕ^4 interaction

Consider then the following interaction contribution to the hamiltonian

$$\hat{H}_{int} = \frac{1}{2} \sum_{q,p,p_1} V(q) \hat{b}_p^\dagger \hat{b}_{p_1}^\dagger \hat{b}_{p+q} \hat{b}_{p-q} \tag{3.46}$$

where $V(r - r_1)$ the potential for two particle interactions, and $V(q)$ its Fourier transform. Noting that the interaction is local in time one finds the general structure for the

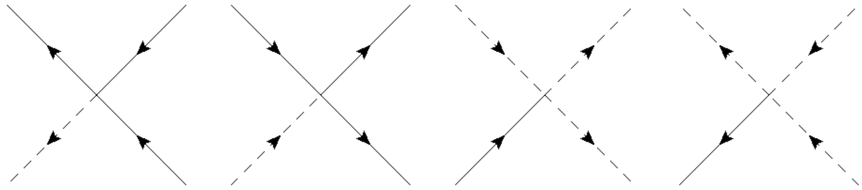

Figure 3.2: Interaction vertices of the ϕ^4-theory. Full lines represent classical components of fields and dashed lines quantum components.

interaction contribution to the actions

$$S_{int}(\phi_+^u, \phi_-^u, \phi_+^d, \phi_-^d) = -\lambda \int dr \int_C dt(\hat{\phi}\phi)^2 = -\lambda \int dr \int_{-\infty}^{\infty} dt[(\phi_+^u \bar{\phi}_+^u)^2 -$$

$$(\phi_-^u \bar{\phi}_-^u)^2 + (\phi_+^d \bar{\phi}_+^d)^2 - (\phi_-^d \bar{\phi}_-^d)^2] = -4\lambda \int dr \int_{-\infty}^{\infty} [\phi^{clu} \bar{\phi}^{clu} \phi^{clu} \bar{\phi}^{qu} +$$

$$+\phi^{clu} \bar{\phi}^{clu} \phi^{clu} \bar{\phi}^{qu} + \phi^{clu} \bar{\phi}^{qu} \phi^{qu} \bar{\phi}^{qu} + \phi^{clu} \bar{\phi}^{qu} \phi^{qu} \bar{\phi}^{qu}] +$$

$$[\phi^{cld} \bar{\phi}^{cld} \phi^{cld} \bar{\phi}^{qd} + \phi^{cld} \bar{\phi}^{cld} \phi^{cld} \bar{\phi}^{qd} + \phi^{cld} \bar{\phi}^{qd} \phi^{qd} \bar{\phi}^{qd} + \phi^{cld} \bar{\phi}^{qd} \phi^{qd} \bar{\phi}^{qd}] \tag{3.47}$$

where in the last line we applied the Keldysh rotation.
It can be verified that inclusion of the ϕ^4 - interaction does not change the normalization condition of the Keldysh partition function $Z = 1$. That is, expanding $e^{S_{int}}$ in powers of V and applying Wick's theorem all contributions are nullified. The reason for this is that ther terms present is the expression are of the type of sum between G^A and G^R at equal times, which are nullified, and there are terms of correlators $\langle \phi^q \phi^q \rangle = 0$ Terms in the last equality of equation (3.47) can diagrammatically be represented as shown in Figure (3.2). These diagrams actually represent only half of all the diagrams appearing, since they are identical for the upper or downer worlds. In order to analyse the self-energy arising from the interaction S_{in} action we take the contractions of interaction terms following a representation until second order pertubation theory for the interaction, as shown in Figures (3.3) and (3.4).

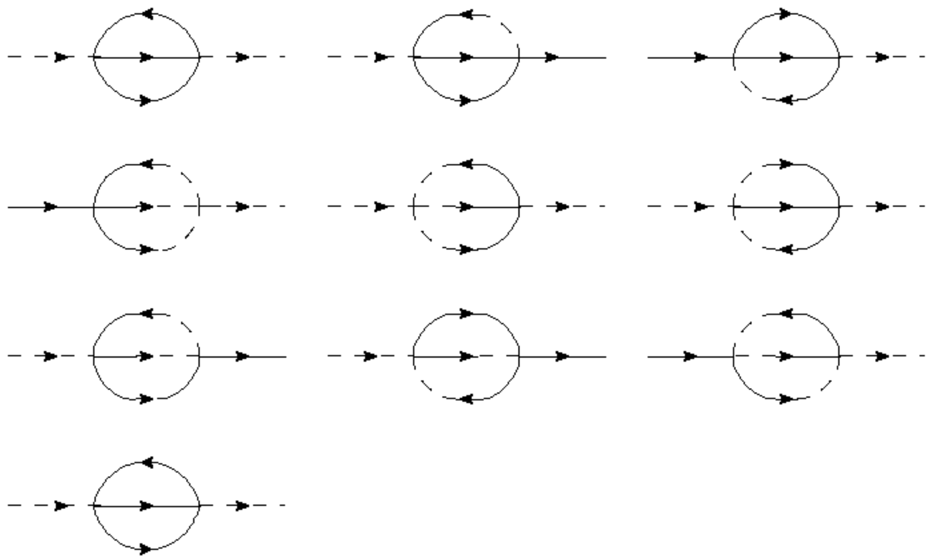

Figure 3.3: Second order self-energy diagrams for ϕ^4-theory: The first line presents interactions in upper or downer worlds, see also first two diagrams in Figure (3.2). The following two lines present interactions in upper or downer worlds, see also diagrams in Figure (3.2). The last line represents the only diagram that describes interactions between the upper and downer worlds.

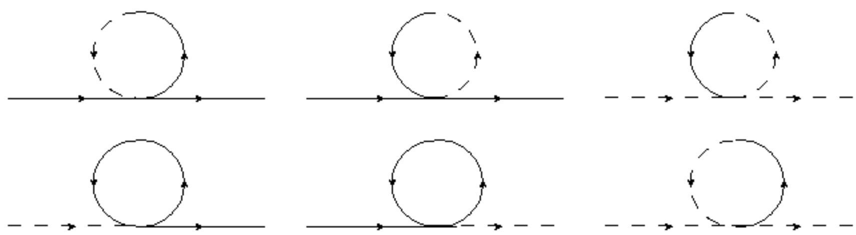

Figure 3.4: One-loop diagrams related to interactions of the first two diagrams in Figure (3.2) with themselves.

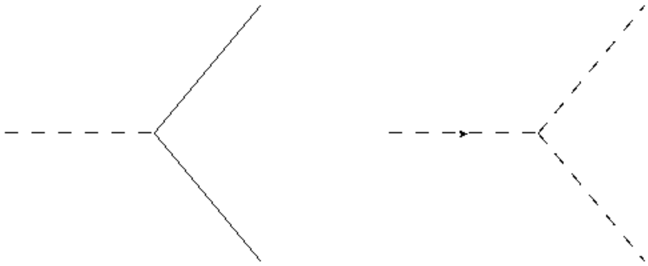

Figure 3.5: Interaction vertices for ϕ^3 theory. Full lines represent classical components of fields and dashed lines quantum components.

Now we calculate the self-energies that arise from these diagrams. The superscript labels the kinds of "legs" of the diagrams. We find

$$\Sigma^{clu(cld)-clu(cld)} = 0$$

$$\Sigma^{clu(cld)-qu(qd)} = \Sigma^{A} = G^{K} - (G^{K})^{2}G^{R}$$

$$\Sigma^{qu(qd)-clu(cld)} = \Sigma^{R} = G^{K} - (G^{K})^{2}G^{A}$$

$$\Sigma^{qu(qd)-qu(qd)} = \Sigma^{K} = -(G^{K})^{3}$$

$$\Sigma^{qu-qd} = \Sigma^{K}_{ud} = -G^{K}_{du}$$

$$\Sigma^{qd-qu} = \Sigma^{K}_{du} = -G^{K}_{ud}.$$

ϕ^3 interaction

Repeating the above procedure for the ϕ^3 interaction we find:

$$S_{int} = \kappa \int dr \int_{-\infty}^{+\infty} dt[(\phi^{clu})^{2}\phi^{qu} + \frac{1}{3}(\phi^{qu})^{3} + (\phi^{cld})^{2}\phi^{qd} + \frac{1}{3}(\phi^{qd})^{3}]. \tag{3.48}$$

The diagrams of these interactions are of two types, as shown in Figure (3.5).

Possible interactions between the two diagrams are given in Figure (3.6).

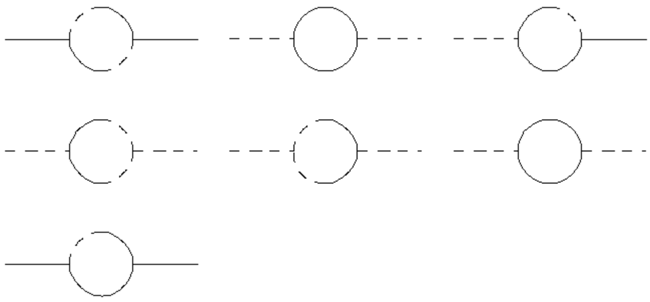

Figure 3.6: Self-energy diagrams for ϕ^3-theory.

Each diagram represents possible interactions of Figure (3.5). As before, the components of the self-energies are found as:

$$\Sigma^{clu(cld)-clu(cld)} = 0$$

$$\Sigma^{clu(cld)-qu(qd)} = \Sigma^A = i\kappa^2 G^K G^A$$

$$\Sigma^{qu(qd)-clu(cld)} = \Sigma^R = i\kappa^2 G^K G^R$$

$$\Sigma^{qu(qd)-qu(qd)} = \Sigma^K = i\kappa^2((G^K)^2 + \frac{1}{3}(G^R)^2 + \frac{1}{3}(G^A)^2)$$

$$\Sigma^{qu-qd} = \Sigma^K_{ud} = i\kappa^2 (G^K_{du})^2$$

$$\Sigma^{qd-qu} = \Sigma^K_{du} = i\kappa^2 (G^K_{ud})^2$$

These self energies are the essential ingredients we need for the derivation of the kinetic equations. They store on the one hand information on relaxation processes and thermalization that determine the conventional distribution functions. That is because when considering the dressed Green's functions, that is, the ones already with the interactions, the imagirany part of the self energies takes the place of the inverse lifetime of the quasi-particles associated to the propagators. On the other hand, within the augmented Keldysh formalism they also store the information on the quantum butterfly effect in form of an instability in the kinetic equation for the new distribution functions parametrizing G_K^{ud} and G_K^{du}.

3.2 Kinetic Equations

We can now expand the Green's functions in powers of the self-energies:

$$\hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + ... \tag{3.49}$$

which can be rewritten in the compact form

$$(\hat{G}_0^{-1} - \hat{\Sigma}) \circ \hat{G} = 1 \tag{3.50}$$

known as the Dyson Equation for the self-energy. With this in mind we are able to calculate the Kinetic Equations for this system. Using the known Green's functions and using self-energy components, i.e. of the last interaction example, the ϕ^3 interaction, we find:

$$\left[\begin{pmatrix} 0 & (G_0^A)^{-1} & 0 & 0 \\ (G_0^R)^{-1} & (G_0^K)^{-1} & 0 & (G_0^{udK})^{-1} \\ 0 & 0 & 0 & (G_0^A)^{-1} \\ 0 & (G_0^{duK})^{-1} & (G_0^R)^{-1} & (G_0^K)^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \Sigma^A & 0 & 0 \\ \Sigma^R & \Sigma^K & 0 & \Sigma^{udK} \\ 0 & 0 & 0 & \Sigma^A \\ 0 & \Sigma^{duK} & \Sigma^R & \Sigma^K \end{pmatrix}\right]$$

$$\circ \begin{pmatrix} G^K & G^R & G^{udK} & 0 \\ G^A & 0 & 0 & 0 \\ G^{duK} & 0 & G^K & R \\ 0 & 0 & G^A & 0 \end{pmatrix} = \hat{I} \tag{3.51}$$

From this matrix equation we can derive the following relations after some algebraic work:

$$((G^{A(R)})^{-1} - \Sigma^{A(R)}) \circ G^{A(R)} = \delta(t - t')\delta(r - r') \tag{3.52}$$

$$[F, (i\partial_t + \frac{1}{2m}\partial_r^2)] = \Sigma^K - (\Sigma^R \circ F - F \circ \Sigma^A) \tag{3.53}$$

$$((G_0^{duK})^{-1} - \Sigma^{duK}) \circ G^A + ((G_0^R)^{-1} - \Sigma^R) \circ G^{duK} = 0 \tag{3.54}$$

$$((G_0^{udK})^{-1} - \Sigma^{udK}) \circ G^A + ((G_0^R)^{-1} - \Sigma^R) \circ G^{udK} = 0 \tag{3.55}$$

Another way to get to the kinetic equations is to calculate the commutator

$$[\hat{G}_0^{-1}, \hat{\Sigma}] = [\hat{\Sigma}, \hat{G}] \tag{3.56}$$

Despite the fact that the fermionic matrices have a different structure, the fermionic

Dyson equations have the same structure, and so does the kinetic equations. This follows from solving the equation below:

$$\left[\begin{pmatrix} (V_0^R)^{-1} & (V_0^K)^{-1} & 0 & (V_0^{udK})^{-1} \\ 0 & (V_0^A)^{-1} & 0 & 0 \\ 0 & (V_0^{duK})^{-1} & (V_0^R)^{-1} & (V_0^K)^{-1} \\ 0 & 0 & 0 & (V_0^A)^{-1} \end{pmatrix} - \begin{pmatrix} \Sigma^R & \Sigma^K & 0 & \Sigma^{udK} \\ 0 & \Sigma^A & 0 & 0 \\ 0 & \Sigma^{duK} & \Sigma^R & \Sigma^K \\ 0 & 0 & 0 & \Sigma^A \end{pmatrix}\right]$$

$$\circ \begin{pmatrix} V^R & V^K & 0 & V^{udK} & \\ 0 & V^A & 0 & 0 & \\ 0 & V^{duK} & V^R & V^K & \\ 0 & 0 & 0 & & V^A \end{pmatrix} = \hat{I} \tag{3.57}$$

We next wish to construct a bridge between the macroscopic and the microscopic domains by connecting the operators commutators to functions on phase-space. In order to achieve this we perform a Wigner transformation in the Dyson equation above. This allow us to consider new coordinates appropriated to analyse the system, conecting microscopic quantities with macroscopic phenomena [19]. We shall use the following relation for the Wigner transformation for the product of two functions A and B,

$$C = A \circ B \overset{WT}{\rightarrow} C = AB + \frac{i}{2}(\partial_x A \partial_p B - \partial_p A \partial_x B) + ...$$

$$[A, B] \overset{WT}{\rightarrow} i(\partial_x A \partial_p B - \partial_p A \partial_x B) + ...$$

Applying these results to the Dyson equation we find the following equations

$$[(1 - \partial_\epsilon Re\Sigma^R)\partial_t + (\partial_t \tilde{V})\partial_\epsilon + \tilde{v}_k \nabla_r - (\nabla_r \tilde{V})\nabla_k]F = \mathcal{I}[F_\epsilon]$$

$$\mathcal{I}[F_\epsilon] = i(\Sigma^K(x,p) - F(x,p)[\Sigma^R(x,p) - \Sigma^A(x,p)])\Delta G_p \tag{3.58}$$

where

$$\tilde{V}(x,p) = V^{cl}(x) + Re\Sigma^R(x,p); \quad \tilde{v}_k = \nabla_k(\omega_k + Re\Sigma^R)$$

where the left hand side holds the kinetic terms, and the right hand side the so-called collision integral, with fixed terms to the self-energies and distribution functions

Next, we have the components corresponding to the Green's functions that connects the upper to the downer worlds and vice-versa. For the downer-to-upper we find

$$((1+\partial_\epsilon Re\Sigma^R)\partial_t + \partial_t(V^{cl} - Re\Sigma^R)\partial_\epsilon - \nabla_k Re\Sigma^R - \nabla_r(V^{cl} - Re\Sigma^R)\nabla_k)F^{du}$$

$$= \mathcal{I}^{du}[F^{du}_\epsilon] = i(\Sigma^{duK}(x,p) - F(x,p)[\Sigma^R(x,p) - \Sigma^A(x,p)] + [\Sigma^R(x,p) - \Sigma^A(x,p)])\Delta G_p$$

$$= i\Sigma^{duK}(x,p) - F^{du}(x,p)[\Sigma^R(x,p) - \Sigma^A(x,p)], \tag{3.59}$$

with $F^{du}(x,p) = F(x,p) - 1$

and for the upper-to-downer we find

$$((1-\partial_\epsilon Re\Sigma^R)\partial_t + \partial_t(V^{cl} + Re\Sigma^R)\partial_\epsilon + \nabla_k Re\Sigma^R - \nabla_r(V^{cl} + Re\Sigma^R)\nabla_k)F^{ud}$$

$$= \mathcal{I}^{ud}[F^{ud}_\epsilon] = i(\Sigma^{udK}(x,p) - F(x,p)[\Sigma^R(x,p) - \Sigma^A(x,p)] - [\Sigma^R(x,p) - \Sigma^A(x,p)])\Delta G_p$$

$$= i\Sigma^{udK}(x,p) - F^{ud}(x,p)[\Sigma^R(x,p) - \Sigma^A(x,p)], \tag{3.60}$$

with $F^{ud}(x,p) = F(x,p) + 1$.

We have now established all the ingredients to study the manifestation of the quantum butterfly effect in a many-body chaotic system. These equations are the ones that describe all the processes concerning the fermionic system. On the left hand side we have the kinetic term, with variations of the distribution functions, while on the right hand side we have the collision term. This one contains the information about the processes of the system, that is, a term where the information about relaxation processes and instabilities processes occur, usually called the ingoing processes and the outgoing processes of the interactions.

The first of these three equations is a well known kinetic equation having a usual distribution function, while the following two represents the ones arising from this new approach, these are represented by new distribution functions concerning the evolution of the system. When we are considering our system in equilibrium is expected, by construction, that the kinetic equations should vanish. This can be seen with the help of some relations between bosonic and fermionic distribution functions. But for the new kinetic equations, to achieve such a nullification in equilibrium we should make use of the relations between the new distribution functions and the old one. With this it is possible to transform the new kinetic equations in the old ones, but as was said, this is supposed to be used only in equilibrium. In the next chapter we will explore this for the weakly interacting electron gas.

Chapter 4

Application: Weakly interacting electrons gas

Let us now apply the formalism developed in the previous chapters to study many-body chaos in a weakly interacting electron gas. We start out from the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where the non-interacting part

$$H_0 = \sum_{\sigma} \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k},\sigma} \tag{4.1}$$

with $c^{\dagger}_{\mathbf{k},\sigma}$ and $c_{\mathbf{k},\sigma}$ the electron creation and annihilation operators with dispersion relation $\epsilon_{\mathbf{k}}$ and σ denotes the spin. The interaction contribution reads

$$H_{int} = -\frac{1}{2} \sum_{\mathbf{q},\mathbf{p},\mathbf{p}_1} V(\mathbf{q}) c^{\dagger}_{\mathbf{p},\sigma} c^{\dagger}_{\mathbf{p}_1,\sigma'} c_{\mathbf{p}_1+\mathbf{q},\sigma'} c_{\mathbf{p}-\mathbf{q},\sigma} \tag{4.2}$$

A diagrammatic representation is given in Figure (4.1).
Building on the discussion of the previous chapter, the action entering the partition function for the augmented Keldysh contour takes the form $S = S_0 + S_{int}$ with

$$S_0 = \int dt \int dr \bar{\psi}(r,t) \hat{G}_0^{-1} \psi(r,t) \tag{4.3}$$

and the interaction contribution, dropping the spin dependence for this analysis, along the augmented Keldysh (aK) contour

$$S_{int} = -\frac{1}{2} \int_{aK} dt \sum_{q,p,p_1} V(q) \bar{\psi}_p \bar{\psi}_{p_1} \psi_{p_1+q} \psi_{p-q} \tag{4.4}$$

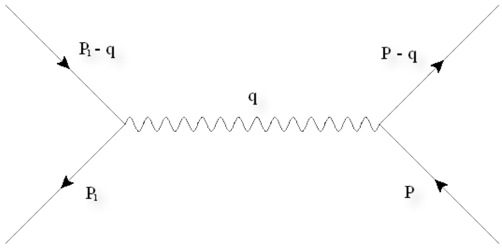

Figure 4.1: Interaction vertex for the electron-electron interaction.

Here G is the Green's function matrix for the fermionic propagator as discussed in the previous chapters,

$$i\hat{G} = \begin{bmatrix} G^K & G^R & G^K_{ud} & 0 \\ G^A & 0 & 0 & 0 \\ G^K_{du} & 0 & G^K & G^R \\ 0 & 0 & G^A & 0 \end{bmatrix} \tag{4.5}$$

We next want to study the manifestation of many-body chaos in this system and derive the Lyapunov exponent characterizing the quantum butterfly effect. To this end we will proceed as follows: We first derive in section 4.1. the usual kinetic equation for the Fermi distribution function. All the information is here stored within the the Keldysh Green's function G^K that lives within "one world" and coupling of the upper and lower worlds does not influence these kinetic equations. We therefore will call these kinetic equations the "intra-world" kinetic equations. In section 4.2 we then turn to the main equations of interest. That is, the generalized kinetic equations that describe the new distribution functions characterizing Keldysh Green's functions G^K_{ud} and G^K_{du} that live between the two worlds. We refer to these equations as the "interworld" kinetic equations. These store information about the two worlds and the quantum butterfly effect. Specifically, we will see how a coupling of the two worlds becomes unstable as a manifestation of the latter. The main task in sections 4.1 and 4.2 is to derive the collision integrals which encode all the information on relaxation and the instability in cases of the intraworld and interworld kinetic equations, respectively. To do this we build on the concepts derived in the previous chapter, i.e. calculate perturbatively the self energies and then perform the Wigner transformation.

In section 4.3 we briefly discuss the equilibrium solutions for the inter- and intraworld kinetic equations. The main idea is then, in the following section 4.4 to study small perturbations around these equilibrium solutions. This will be done by linearizing the collision integrals in the small perturbation, and we will do this separately for the intraworld (section 4.4.1) and interworld (section 4.4.2) equations. In the final section 4.5 we then diagonalize the linearized intraworld kinetic equations and show that the linearized collision operator posses positive eigenvalues that lead to an exponential blow up of small perturbations around the equilibrium solution. This is very different from the linearized interworld kinetic equation where it is known that the spectrum of the linearized collision operator consists only of negative eigenvalues which describe the rates on which perturbations around equilibrium solutions relax. This instability in the intraworld collision operator manifests the quantum butterfly effect and the positive eigenvalue set the possible Lyapunov exponent.

4.1 Intraworld kinetic equations

From the previous chapter we know that it is interesting to calculate the kinetic equations of the physical system. In order to achieve this we need to calculate the self-energies for the electron interactions by expansion of the interaction term to second order.
Before we turn to the explicit calculation, we briefly comment on the “left hand side" of the kinetic equations achieved in the previous chapter. The left hand side of the kinetic equations have some derivatives that we analyse. First, we can neglect the terms with spatial derivatives since we shall consider here a spatial homogeneous system. That makes all derivatives with respect to **r** null.

The mass-shell approximation considers the relation between the sharp peaked difference $G^R - G^A$ and the slow growing distribution function $F(\mathbf{r}, t, \mathbf{k}, \tilde{\epsilon})$ for an $\tilde{\epsilon} = \epsilon - \omega_k - V^{cl} + Re\Sigma^R$. Since, as already known by the structure of the Keldysh Green's function, G^K the distribution function always shows up in a product of $G^R - G^A$ by $F(\mathbf{r}, t, \mathbf{k}, \epsilon)$, but to make progress away from equilibrium we should use $F(\mathbf{r}, t, \mathbf{k}, \tilde{\epsilon})$. But it happens that $G^R - G^A$ is peaked, delta function, in $\tilde{\epsilon}$, so not beeing F dependent on $\tilde{\epsilon}$. As long as the characteristic energy scale $\delta\tilde{\epsilon}$ of the referred distribution function is much larger than the inverse quasiparticle lifetime it is possible to neglect the $\tilde{\epsilon}$ [18]. The real part of the self-energy presence in the kinetic term can be neglected as a derivative of the energy in comparison with the unity forming the term weighting the time derivative. All the previous facts allows to approximate the kinetic term on the left hand side by the simple form $\partial_t \tilde{F} = \mathcal{I}$.

The "right hand side" of the kinetic equations is the more interesting. It is non-linear in the distribution functions since it accounts for the particle collisions that play the important role of a bath to the system itself [18]. The masss-shell formalism then works like a semi-classical approximation. We have that the distribution function works as a classical object to be seeing as a probability distribution, and the collision integral incorporate non-linear traces of local interactions of the many-body system. The general forms of collision integrals entering the kinetic equations were already calculated in the previous chapter and we can build on these results

$$\mathcal{I}[F_\epsilon] = i[G_p\Sigma_p - \Sigma_p G_p]^K = i\Sigma_p^K[G_p^R - G_p^A] - i[\Sigma_p^R - \Sigma_p^A]G_p^K =$$

$$= i\left(\Sigma_p^K - [\Sigma_p^R - \Sigma_p^A]F_p\right)\Delta G_p, \tag{4.6}$$

where we use four-vector notation $p = (\epsilon, \epsilon_{\mathbf{p}})$ and $\Delta G_p = G_p^R - G_p^A = -2\pi i\delta(\epsilon - \epsilon_{\mathbf{k}})$ fixes energies on-shell. We next have to specify the self energies for the problem at hand. We will do this perturbatively up to second order. We take into account two important contributions known as the "direct" and the "exchange" contribution.

4.1.1 Self-energy

We now calculate perturbatively the self-energy for the bare electron-electron interactions $V_q = V(\mathbf{q})$. Since this is an instantaneous interaction we notice that $V^R = V^A$ and $V^K = 0$, because of its structure of $V^R - V^A$, which simplifies the calculations.

First order

For notational convenience we here suppress summations and integrals. Since we are here focusing on the up and down cases isolated, the indices belong all to one world and $a, b = cl, q$ denotes only the Keldysh component. The self-energy to first order reads

$$-2i\Sigma = \hat{\gamma}^a\langle\psi\bar{\psi}\rangle\hat{\gamma}^b \times \langle\phi^b\phi^a\rangle = (\hat{\gamma}^a\hat{G}\hat{\gamma}^b) \times \hat{V}^{ba}$$

$$= V_q^K\,\gamma^{cl}G_{p-q}\gamma^{cl} + V_q^A\,\gamma^{cl}G_{p-q}\gamma^q + V_q^R\,\gamma^q G_{p-q}\gamma^{cl}, \tag{4.7}$$

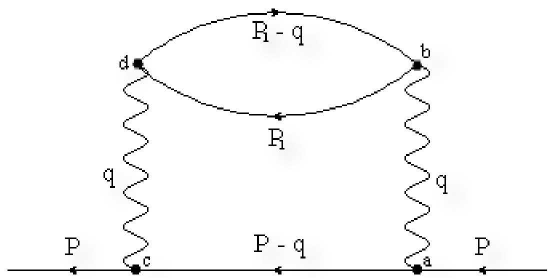

Figure 4.2: Diagrammatic representation of the direct contribution to the self-energy. Note that its is the polarization bubble that gives rises to retardation-effects in the interaction.

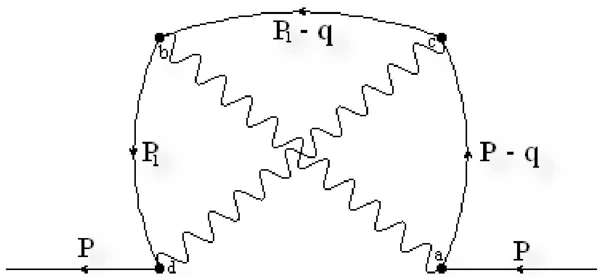

Figure 4.3: Diagrammatic representation of the exchange contribution to the self-energy.

or explicitly

$$-2i\Sigma_p^R = V_q^R G_{p-q}^K + V_q^K G_{p-q}^A, \tag{4.8}$$

$$-2i\Sigma_p^A = V_q^A G_{p-q}^K + V_q^K G_{p-q}^R, \tag{4.9}$$

$$-2i\Sigma_p^K = V_q^R G_{p-q}^A + V_q^A G_{p-q}^R + V_q^K G_{p-q}^K. \tag{4.10}$$

To first order the collision integral for instantaneous interaction vanishes.

Second order

In second order we have now the direct and exchange contributions. In terms of Feynman diagrams these two contributions correspond to two possible connections between interaction lines (4.1) as shown in Figures (4.2) and (4.3). Use the diagrams to understand the construction of the self-energies equations structure.

Direct term: Starting with the former,

$$
\begin{aligned}
-2i\Sigma_p &= \langle \phi_q^a \phi_{-q}^b \rangle \langle \phi_{-q}^c \phi_q^d \rangle \left[\gamma^a G_{p-q} \gamma^c\right] \operatorname{tr}\left(\gamma^b G_{p_1} \gamma^d G_{p_1-q}\right) \\
&= V_q^R V_q^A \gamma^{cl} G_{p-q} \gamma^{cl} \operatorname{tr}\left(\gamma^q G_{p_1} \gamma^q G_{p_1-q}\right) \\
&+ V_q^A V_q^A \gamma^q G_{p-q} \gamma^{cl} \operatorname{tr}\left(\gamma^{cl} G_{p_1} \gamma^q G_{p_1-q}\right) \\
&+ V_q^R V_q^R \gamma^{cl} G_{p-q} \gamma^q \operatorname{tr}\left(\gamma^q G_{p_1} \gamma^{cl} G_{p_1-q}\right)
\end{aligned} \tag{4.11}
$$

To explicitly calculate this it is interesting to notice that for a given square matrix A, the product $\gamma^{cl} A \gamma^{cl}$ will keep the A matrix invariant. The product $\gamma^{cl} A \gamma^q$ changes the columns by a transformation with vertical symmetry. The product $\gamma^q A \gamma^{cl}$ rotates vertically with the symmetry around the horizontal medium of the matrix. The product $\gamma^q A \gamma^q$ performs the previous two transformation sequentially. Now, using the instantaneous interaction potential we have for the components of the self-energy (knowing that the self-energy has the same matrix structure as the G_0^{-1}):

$$
\begin{aligned}
-2i\Sigma^K = V_q^2 (& G_{p_1}^K G_{p_1-q}^K G_{p-q}^K + G_{p_1}^R G_{p_1-q}^A G_{p-q}^K + G_{p_1}^A G_{p_1-q}^R G_{p-q}^K + \\
&+ G_{p_1}^K G_{p_1-q}^R G_{p-q}^A + G_{p_1}^A G_{p_1-q}^K G_{p-q}^A + G_{p_1}^R G_{p_1-q}^K G_{p-q}^R + G_{p_1}^K G_{p_1-q}^A G_{p-q}^R)
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
-2i\Sigma^R = -V_q^2 (& G_{p_1}^A G_{p_1-q}^R G_{p-q}^R + G_{p_1}^K G_{p_1-q}^K G_{p-q}^R + G_{p_1}^R G_{p_1-q}^A G_{p-q}^R + \\
& G_{p_1}^R G_{p_1-q}^K G_{p-q}^K + G_{p_1}^K G_{p_1-q}^A G_{p-q}^K)
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
-2i\Sigma^A = V_q^2 (& G_{p_1}^A G_{p_1-q}^R G_{p-q}^A + G_{p_1}^K G_{p_1-q}^K G_{p-q}^A + G_{p_1}^R G_{p_1-q}^A G_{p-q}^A + \\
& G_{p_1}^K G_{p_1-q}^R G_{p-q}^K + G_{p_1}^A G_{p_1-q}^K G_{p-q}^K).
\end{aligned} \tag{4.14}
$$

We can write in a more interesting way the equations, with the use of $G_p^K = F_p \Delta G_p$:

$$
-2i\Sigma_q^K = V_q^2 (F_{p_1} F_{p_1-q} F_{p-q} - F_{p-q} - F_{p_1} + F_{p_1-q}) \Delta G_{p_1-q} \Delta G_{p_1} \Delta G_{p-q} \tag{4.15}
$$

$$
-2i(\Sigma_q^R - \Sigma_q^A) = V_q^2 (F_{p_1} F_{p_1-q} - 1 + F_{p-q}(F_{p_1-q} - F_{p_1})) \Delta G_{p_1-q} \Delta G_{p_1} \Delta G_{p-q} \tag{4.16}
$$

where we used causality to rewrite $G^R G^A + G^A G^R = -\Delta G$.

Now, with the structure of the collision integral (4.6) we have for the direct channel in the Keldysh space:

$$\begin{aligned}\mathcal{I}[F_\epsilon] &= \left(\Sigma_p^K - [\Sigma_p^R - \Sigma_p^A]F_p\right) \\ &= \frac{1}{2}V_q^2\left([F_{p_1}F_{p_1-q}-1][F_{p-q}-F_p] - [F_pF_{p-q}-1][F_{p_1-q}-F_{p_1}]\right)\times \\ &\times \Delta G_{p_1}\Delta G_{p_1-q}\Delta G_{p-q}\end{aligned} \tag{4.17}$$

Exchange term:

$$\begin{aligned}&-2i\Sigma_p = \langle\phi_q^a\phi_{-q}^b\rangle\langle\phi_{p-p_1}^c\phi_{p_1-p}^d\rangle\,\gamma^aG_{p-q}\gamma^cG_{p_1-q}\gamma^bG_{p_1}\gamma^d \\ &= V_q^RV_{p-p_1}^R\gamma^{cl}G_{p-q}\gamma^{cl}G_{p_1-q}\gamma^qG_{p_1}\gamma^q + V_q^RV_{p-p_1}^A\gamma^{cl}G_{p-q}\gamma^qG_{p_1-q}\gamma^qG_{p_1}\gamma^{cl} \\ &+ V_q^AV_{p-p_1}^A\gamma^qG_{p-q}\gamma^qG_{p_1-q}\gamma^{cl}G_{p_1}\gamma^{cl}\end{aligned} \tag{4.18}$$

$$\begin{aligned}&= V_q^RV_{p-p_1}^R\begin{pmatrix} G_{p-q}^RG_{p_1-q}^KG_{p_1}^K + G_{p-q}^KG_{p_1-q}^AG_{p_1}^K & G_{p-q}^RG_{p_1-q}^KG_{p_1}^R + G_{p-q}^KG_{p_1-q}^AG_{p_1}^R \\ 0 & 0\end{pmatrix} \\ &+ V_q^RV_{p-p_1}^A\begin{pmatrix} G_{p-q}^RG_{p_1-q}^AG_{p_1}^R + G_{p-q}^KG_{p_1-q}^KG_{p_1}^R & G_{p-q}^RG_{p_1-q}^AG_{p_1}^K + G_{p-q}^KG_{p_1-q}^KG_{p_1}^K + G_{p-q}^KG_{p_1-q}^RG_{p_1}^A \\ 0 & G_{p-q}^AG_{p_1-q}^KG_{p_1}^K + G_{p-q}^AG_{p_1-q}^RG_{p_1}^A\end{pmatrix} \\ &+ V_q^AV_{p-p_1}^A\begin{pmatrix} 0 & G_{p-q}^AG_{p_1-q}^RG_{p_1}^K + G_{p-q}^AG_{p_1-q}^KG_{p_1}^A \\ 0 & G_{p-q}^KG_{p_1-q}^RG_{p_1}^K + G_{p-q}^KG_{p_1-q}^KG_{p_1}^A\end{pmatrix}\end{aligned} \tag{4.19}$$

and therefore

$$\begin{aligned}-2i\Sigma_p^K &= V_q^RV_{p-p_1}^R\left(G_{p-q}^RG_{p_1-q}^KG_{p_1}^R + G_{p-q}^KG_{p_1-q}^AG_{p_1}^R\right) \\ &\quad + V_q^RV_{p-p_1}^A\left(G_{p-q}^RG_{p_1-q}^AG_{p_1}^K + G_{p-q}^KG_{p_1-q}^KG_{p_1}^K + G_{p-q}^KG_{p_1-q}^RG_{p_1}^A\right) \\ &\quad + V_q^AV_{p-p_1}^A\left(G_{p-q}^AG_{p_1-q}^RG_{p_1}^K + G_{p-q}^AG_{p_1-q}^KG_{p_1}^A\right) \\ &= V_qV_{p-p_1}(G_{p-q}^KG_{p_1-q}^KG_{p_1}^K - \Delta G_{p-q}\Delta G_{p_1-q}G_{p_1}^K - G_{p-q}^K\Delta G_{p_1-q}\Delta G_{p_1} + \\ &\quad + \Delta G_{p-q}G_{p_1-q}^K\Delta G_{p_1}) \\ &= V_qV_{p-p_1}\left(F_{p-q}F_{p_1-q}F_{p_1} - F_{p_1} - F_{p-q} + F_{p_1-q}\right)\Delta G_{p-q}\Delta G_{p_1-q}\Delta G_{p_1}\end{aligned} \tag{4.20}$$

where in the second equality we use that the interaction is instantaneous.

Similarly,

$$
\begin{aligned}
-2i(\Sigma_p^R - \Sigma_p^A) = & V_q^R V_{p-p_1}^R \left(G_{p-q}^R G_{p_1-q}^K G_{p_1}^K + G_{p-q}^K G_{p_1-q}^A G_{p_1}^K \right) \\
& + V_q^R V_{p-p_1}^A (G_{p-q}^R G_{p_1-q}^A G_{p_1}^R + G_{p-q}^K G_{p_1-q}^K G_{p_1}^R - G_{p-q}^A G_{p_1-q}^K G_{p_1}^K - \\
& - G_{p-q}^A G_{p_1-q}^R G_{p_1}^A) \\
& - V_q^A V_{p-p_1}^A \left(G_{p-q}^K G_{p_1-q}^R G_{p_1}^K + G_{p-q}^K G_{p_1-q}^K G_{p_1}^A \right) \\
= & V_q V_{p-p_1} (\Delta G_{p-q} G_{p_1-q}^K G_{p_1}^K - G_{p-q}^K \Delta G_{p_1-q} G_{p_1}^K + G_{p-q}^K G_{p_1-q}^K \Delta G_{p_1} - \\
& - \Delta G_{p-q} \Delta G_{p_1-q} \Delta G_{p_1}) \\
= & V_q V_{p-p_1} \left(F_{p_1-q} F_{p_1} - F_{p-q} F_{p_1} + F_{p-q} F_{p_1-q} - 1 \right) \Delta G_{p-q} \Delta G_{p_1-q} \Delta G_{p_1}
\end{aligned}
\tag{4.21}
$$

where in the second equality used that $G^R G^A G^R - G^A G^R G^A = -\Delta G \Delta G \Delta G$. Thus, we obtain

$$
\begin{aligned}
\mathcal{I}[F_\epsilon] = & \left(\Sigma_p^K - [\Sigma_p^R - \Sigma_p^A] F_p \right) \\
= & -\frac{1}{2} V_q V_{p-p_1} \left([F_{p_1} F_{p_1-q} - 1][F_{p-q} - F_p] - [F_p F_{p-q} - 1][F_{p_1-q} - F_{p_1}] \right) \times \\
& \times \Delta G_{p-q} \Delta G_{p_1-q} \Delta G_{p_1} \Delta G_p
\end{aligned}
\tag{4.22}
$$

Now we notice that the collision integrals from the direct channel and exchange channel are equal, except for the potential and signs. We compute a total collision integral for the second order expansion of the electron-electron interaction summing the contributions for both channels:

$$
\begin{aligned}
\mathcal{I}[F_\epsilon] = & \frac{1}{2} [V_q^2 - V_q V_{p-p_1}] \left([F_{p_1} F_{p_1-q} - 1][F_p - F_{p-q}] + [F_p F_{p-q} - 1][F_{p_1-q} - F_{p_1}] \right) \times \\
& \times \Delta G_{p-q} \Delta G_{p_1-q} \Delta G_{p_1} \Delta G_p
\end{aligned}
\tag{4.23}
$$

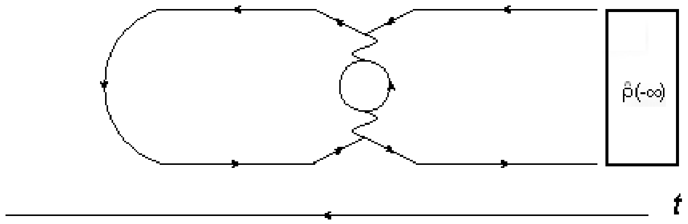

Figure 4.4: Representation of a second order interaction process involving a polarization bubble (the fermion loop in the middle). The presence of a fermion loop introduces retardation to the interaction and therefore allows to have interactions between different worlds. The exchange contribution cannot be represented like this since there is no polarization bubble to give retardation, it is a local interaction in time. Recall that for instantaneous interactions $V_A = V_R$.

4.2 Interworld kinetic equations

In the augmented world formalism the collision integral entering the kinetic equations for the interworld distribution reads as was calculated in equations (3.59) and (3.60)

$$\mathcal{I}[F_p^{ud}] = \left(\Sigma^K_{\epsilon,ud} - [\Sigma^R_\epsilon - \Sigma^A_\epsilon]F^{ud}_\epsilon\right) \tag{4.24}$$

$$\mathcal{I}[F_p^{du}] = \left(\Sigma^K_{\epsilon,du} - [\Sigma^R_\epsilon - \Sigma^A_\epsilon]F^{du}_\epsilon\right) \tag{4.25}$$

It follows that we only need to calculate the self-energy within a single world, $[\Sigma^R_\epsilon - \Sigma^A_\epsilon]$. We only need to calculate the Keldysh component of the interworld self-energies $\Sigma^K_{\epsilon,ud}$ and $\Sigma^K_{\epsilon,ud}$, respectively.

First order

We recall that vertices are local in time (i.e. interaction lines do not connect different worlds) it follows that $\Sigma^K_{ud} = \Sigma^K_{du} = 0$. Recalling further that $\Sigma^R - \Sigma^A \propto (V^R - V^A)G^K = 0$, we notice that the first order contribution to the self-energy vanishes. However, you can see directly by applying in the equation

$$-2i\Sigma = \hat{\gamma}^a\langle\psi\bar{\psi}\rangle\hat{\gamma}^b \times \langle\phi^b\phi^a\rangle = (\hat{\gamma}^a\hat{G}\hat{\gamma}^b) \times \hat{V}^{ba} \tag{4.26}$$

where this time we can represent the γ in a 4×4 version of the matrices, like $\gamma^b = \gamma^a \otimes \hat{I}$. In this way we transform the original 4×4 matrices for the Green's functions with the four blocks, two for the up and down world, and two others for the mix of the down and up. But we could instead use the interworld blocks (up-down and down-up) separately.

The same for the self-energies, where

$$G^{ud} = \begin{pmatrix} 0 & G^K_{ud} \\ 0 & 0 \end{pmatrix} \qquad G^{du} = \begin{pmatrix} 0 & G^K_{du} \\ 0 & 0 \end{pmatrix}. \tag{4.27}$$

Second order

We again consider separately the direct and exchange contributions.

Direct term: Recovering the structure for the second order expression for the self-energy we notice that

$$\begin{aligned} -2i\Sigma^K_{q,ud} &= \langle \phi^a_q \phi^b_{-q} \rangle \langle \phi^c_{-q} \phi^d_q \rangle \left[\gamma^a G_{p-q} \gamma^c\right] \mathrm{tr}\left(\gamma^b G_{p_1} \gamma^d G_{p_1-q}\right) \\ &= V^R_q V^A_q \gamma^{clu} G_{p-q} \gamma^{cld} \mathrm{tr}\left(\gamma^{qd} G_{p_1} \gamma^{qu} G_{p_1-q}\right) \\ &= V^2_q F^{ud}_{p-q} F^{ud}_{p_1} F^{du}_{p_1-q} \Delta G_{p-q} \Delta G_{p_1} \Delta G_{p_1-q}, \end{aligned} \tag{4.28}$$

$$\begin{aligned} -2i\Sigma^K_{q,ud} &= \langle \phi^a_q \phi^b_{-q} \rangle \langle \phi^c_{-q} \phi^d_q \rangle \left[\gamma^a G_{p-q} \gamma^c\right] \mathrm{tr}\left(\gamma^b G_{p_1} \gamma^d G_{p_1-q}\right) \\ &= V^R_q V^A_q \gamma^{cld} G_{p-q} \gamma^{clu} \mathrm{tr}\left(\gamma^{qu} G_{p_1} \gamma^{qd} G_{p_1-q}\right) \\ &= V^2_q F^{du}_{p-q} F^{du}_{p_1} F^{ud}_{p_1-q} \Delta G_{p-q} \Delta G_{p_1} \Delta G_{p_1-q}. \end{aligned} \tag{4.29}$$

Analogously we recall Equation (4.16) ,

$$-2i(\Sigma^R_p - \Sigma^A_p) = V^2_q \left([F_{p_1} F_{p_1-q} - 1] + [F_{p_1-q} - F_{p_1}] F_{p-q}\right) \Delta G_{p_1} \Delta G_{p_1-q} \Delta G_{p-q}. \tag{4.30}$$

That is,

$$\begin{aligned} \mathcal{I}[F^{ud}_\epsilon] &= \left(\Sigma^K_{q,ud} - [\Sigma^R_p - \Sigma^A_p] F^{ud}_p\right) \\ &= V^2_q \left(F^{ud}_{p-q} F^{ud}_{p_1} F^{du}_{p_1-q} - \{[F_{p_1} F_{p_1-q} - 1] + [F_{p_1-q} - F_{p_1}] F_{p-q}\} F^{ud}_p\right) \\ &\quad \times \Delta G_{p-q} \Delta G_{p_1} \Delta G_{p_1-q}, \end{aligned} \tag{4.31}$$

and similarly,

$$\mathcal{I}[F_\epsilon^{du}] = \left(\Sigma_{q,du}^K - [\Sigma_p^R - \Sigma_p^A]F_p^{du}\right)$$
$$= V_q^2 \left(F_{p-q}^{du}F_{p_1}^{du}F_{p_1-q}^{ud} - \{[F_{p_1}F_{p_1-q} - 1] + [F_{p_1-q} - F_{p_1}]F_{p-q}\}\, F_p^{du}\right)$$
$$\times \Delta G_{p-q}\Delta G_{p_1}\Delta G_{p_1-q}. \tag{4.32}$$

Exchange term: Similarly as the argumentation above for the collision integral, for the exchange contribution we can use the results of the intraworld self-energies:

$$\begin{aligned} -2i\Sigma_{q,ud}^K &= \langle\phi_q^a\phi_{-q}^b\rangle\langle\phi_{p-p_1}^c\phi_{p_1-p}^d\rangle\, \gamma^a G_{p-q}\gamma^c G_{p_1-q}\gamma^b G_{p_1}\gamma^d \\ &= V_q^R V_{p-p_1}^A \gamma^{clu} G_{p-q}\gamma^{qu} G_{p_1-q}\gamma^{qd} G_{p_1}\gamma^{cld} \\ &= V_q V_{p-p_1} G_{p-q,ud}^K G_{p_1-q,du}^K G_{p_1,ud}^K \\ &= V_q V_{p-p_1} F_{p-q}^{ud} F_{p_1-q}^{du} F_{p_1}^{ud} \Delta G_{p-q}\Delta G_{p_1-q}\Delta G_{p_1}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} -2i\Sigma_{q,du}^K &= \langle\phi_q^a\phi_{-q}^b\rangle\langle\phi_{p-p_1}^c\phi_{p_1-p}^d\rangle\, \gamma^a G_{p-q}\gamma^c G_{p_1-q}\gamma^b G_{p_1}\gamma^d \\ &= V_q^R V_{p-p_1}^A \gamma^{cld} G_{p-q}\gamma^{qd} G_{p_1-q}\gamma^{qu} G_{p_1}\gamma^{clu} \\ &= V_q V_{p-p_1} G_{p-q,du}^K G_{p_1-q,ud}^K G_{p_1,du}^K \\ &= V_q V_{p-p_1} F_{p-q}^{du} F_{p_1-q}^{ud} F_{p_1}^{du} \Delta G_{p-q}\Delta G_{p_1-q}\Delta G_{p_1}, \end{aligned} \tag{4.34}$$

and we recall again equation (4.16)

$$-2i(\Sigma_p^R - \Sigma_p^A) = V_q V_{p-p_1}\left([F_{p_1}F_{p_1-q} - 1] + [F_{p_1-q} - F_{p_1}]F_{p-q}\right)$$
$$\times \Delta G_{p_1}\Delta G_{p_1-q}\Delta G_{p-q}. \tag{4.35}$$

That is,

$$\begin{aligned}\mathcal{I}[F_\epsilon^{ud}] &= \left(\Sigma^K_{q,ud} - [\Sigma^R_p - \Sigma^A_p]F^{ud}_p\right)\\ &= -V_qV_{p-p_1}\left(F^{ud}_{p-q}F^{ud}_{p_1}F^{du}_{p_1-q} - \{[F_{p_1}F_{p_1-q}-1] + [F_{p_1-q}-F_{p_1}]F_{p-q}\}\,F^{ud}_p\right)\\ &\qquad \times \Delta G_{p-q}\Delta G_{p_1}\Delta G_{p_1-q},\end{aligned} \tag{4.36}$$

and similarly,

$$\begin{aligned}\mathcal{I}[F_\epsilon^{du}] &= \left(\Sigma^K_{q,du} - [\Sigma^R_p - \Sigma^A_p]F^{du}_p\right)\\ &= -V_qV_{p-p_1}\left(F^{du}_{p-q}F^{du}_{p_1}F^{ud}_{p_1-q} - \{[F_{p_1}F_{p_1-q}-1] + [F_{p_1-q}-F_{p_1}]F_{p-q}\}\,F^{du}_p\right)\\ &\qquad \times \Delta G_{p-q}\Delta G_{p_1}\Delta G_{p_1-q}.\end{aligned} \tag{4.37}$$

Analogously to the Keldysh space we can see the similarities by adding the collision integrals for the interworld part

$$\begin{aligned}\mathcal{I}[F_p^{ud}] = V_q[V_q - V_{p-p_1}]&\left(F^{ud}_{p-q}F^{ud}_{p_1}F^{du}_{p_1-q} - \{[F_{p_1}F_{p_1-q}-1] + [F_{p_1-q}-F_{p_1}]F_{p-q}\}\,F^{ud}_p\right)\\ &\times \Delta G_{p-q}\Delta G_{p_1}\Delta G_{p_1-q},\end{aligned} \tag{4.38}$$

$$\begin{aligned}\mathcal{I}[F_p^{du}] = V_q[V_q - V_{p-p_1}]&\left(F^{du}_{p-q}F^{du}_{p_1}F^{ud}_{p_1-q} - \{[F_{p_1}F_{p_1-q}-1] + [F_{p_1-q}-F_{p_1}]F_{p-q}\}\,F^{du}_p\right)\\ &\times \Delta G_{p-q}\Delta G_{p_1}\Delta G_{p_1-q}.\end{aligned} \tag{4.39}$$

Having derived the collision integrals for the intra- and interworld kinetic equations we next want to study the information stored within them. Specifically the latter knows about the quantum butterfly effect and we will in the remaining part of this chapter see how to extract from it Lyapunov rates. Before, we have to discuss equilibrium solutions to the kinetic equations.

4.3 Equilibrium solutions

Equilibrium solutions to the kinetic equations are those distributions which do not change in time. Technically, they are those which nullify the collision integrals. In this section we briefly discuss the equilibrium distributions for the intra and interworld kinetic equations.

In the next section we then perturb these equilibrium solutions and derive the linearized collision integrals in the small perturbations.

4.3.1 Intraworld

Recalling the following relation

$$\frac{1 - F_{\epsilon_+} F_{\epsilon_-}}{F_{\epsilon_+} - F_{\epsilon_-}} = B_\omega \tag{4.40}$$

between bosonic, $B(\omega) \equiv \coth(\omega/2T)$, and fermionic, $F(\epsilon) \equiv \tanh(\epsilon/2T)$, distribution function, where $\epsilon_+ = \epsilon + \omega/2$ and $\epsilon_- = \epsilon - \omega/2$, for the equilibrium regime. We can verify that the general collision integral vanishes for the equilibrium Fermi distribution.

$$\begin{aligned}
&[F_{p_1} F_{p_1 - q} - 1][F_{p-q} - F_p] - [F_p F_{p-q} - 1][F_{p_1 - q} - F_{p_1}] \\
&\quad = B_q \left([F_{p_1} - F_{p_1 - q}][F_{p-q} - F_p] + [F_{p-q} - F_p][F_{p_1 - q} - F_{p_1}] \right) = 0
\end{aligned} \tag{4.41}$$

This solution is stable in the sense that small deviations from the equilibrium solutions are suppressed, i.e. inserting $F = F^{\text{eq}} + \delta F$ into the collision integral $\mathcal{I}[F_\epsilon] = M \delta F_\epsilon$ with M a negative definite matrix. We shall see in future calculations this statement. We expect that the eigenvalues of this matrix define a exponential that abruptly goes to zero. In equilibrium the losses and gains due to many particle collision compensate each other, the "In" and "Out" terms compensate each other, then this would be a stable process. We know from the Boltzmann H-Theorem that the interactions between the particles increases entropy to a maximizing state [19]. The structure obtained by the collision integral structure guarantees, by the formal definition of entropy as $S = -\langle ln(f/e) \rangle$, with f being the equilibrium Boltzmann distribution function, that the variation over time of the entropy is positive [19], that is, it increases over time evolution of the particle interactions. We shall see better this by the symmetry of the linearized form of the equations, where we shall expect that the interworld part has not this accordance with the H-Theorem.

4.3.2 Interworld

A trivial equilibrium solution is that of decoupled worlds

$$F^{ud}_p = 0, \qquad F^{du}_p = 0, \tag{4.42}$$

There exists, however, a second non-trivial solution. Assuming that interworld distributions have relaxed to their equilibrium value, we can reorganize the distribution functions entering the collision integral as

$$\mathcal{F}_{p_1}\bar{\mathcal{F}}_{p_1-q}\mathcal{F}_{p-q} - (F_{p_1-q}F_{p_1} - 1)\mathcal{F}_p - (F_{p_1-q} - F_{p_1})F_{p-q}\mathcal{F}_p \tag{4.43}$$

and then notice that inserting the real values of the interworld distribution functions, defined at the end of the second chapter.

$$F^{ud}_p = F_p + 1, \qquad F^{du}_p = F_p - 1, \tag{4.44}$$

thus, we can recover the same expression for the distribution functions for the usual Keldysh world, which we already know that vanishes in equilibrium, as showed in subsection (4.3). The trivial is a stable the non-trivial an unstable solution, leading to the butterfly effect.

4.4 Linearized collision integrals

In this section we study the linearized kinetic equations under conditions which lead to homogeneous distribution functions, i.e.

$$\partial_t F_{\mathbf{p}} = \mathcal{I}[F_{\mathbf{p}}]. \tag{4.45}$$

We first concentrate on the usual intraworld case.

4.4.1 Linearized intraworld kinetic equation

We recall that summations $\sum_p = \int \frac{d\epsilon}{2\pi} \sum_{\mathbf{p}}$ and $\Delta G_p = -2\pi i \delta(\epsilon - \epsilon_{\mathbf{p}})$, and on-shell distribution functions $F_{\mathbf{p}} = \int d\epsilon F_p \delta(\epsilon - \epsilon_{\mathbf{p}})$. The collision integral is then of the form

$$\begin{aligned}
\mathcal{I}[F_{\mathbf{p}}] &= \frac{1}{2} \sum_{q,p_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}] \left([F_{p_1} F_{p_1-q} - 1][F_{p-q} - F_p] - [F_p F_{p-q} - 1][F_{p_1-q} - F_{p_1}] \right) \times \\
&\quad \times \Delta G_{p-q} \Delta G_{p_1-q} \Delta G_{p_1} \Delta G_p \\
&= \frac{1}{2} \sum_{\mathbf{q},\mathbf{p}_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}] \left([F_{\mathbf{p}_1} F_{\mathbf{p}_1-\mathbf{q}} - 1][F_{\mathbf{p}} - F_{\mathbf{p}-\mathbf{q}}] + [F_{\mathbf{p}} F_{\mathbf{p}-\mathbf{q}} - 1][F_{\mathbf{p}_1-\mathbf{q}} - F_{\mathbf{p}_1}] \right) \\
&\delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q}) \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p} - q]) \\
&= -4 \sum_{\mathbf{q},\mathbf{p}_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}] \left(n_{\mathbf{p}} n_{\mathbf{p}_1-\mathbf{q}} n^h_{\mathbf{p}_1} n^h_{\mathbf{p}-\mathbf{q}} - n^h_{\mathbf{p}} n^h_{\mathbf{p}_1-\mathbf{q}} n_{\mathbf{p}_1} n_{\mathbf{p}-\mathbf{q}} \right) \\
&\quad \times \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q}) \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p} - q]) \\
&\equiv \mathcal{I}^{\text{in}}[n_{\mathbf{p}}] - \mathcal{I}^{\text{out}}[n_{\mathbf{p}}],
\end{aligned} \tag{4.46}$$

where in the third but last line we introduced the electron and hole distributions $F_{\mathbf{p}} = 1 - 2n_{\mathbf{p}}$ and $n^h_{\mathbf{p}} = 1 - n_{\mathbf{p}}$, respectively.

We are interested in small deviations around the equilibrium solution. Then, it is convenient to parametrize distributions as

$$n_{\mathbf{p}} = f_{\mathbf{p}} + f_{\mathbf{p}}(1 - f_{\mathbf{p}})\chi_{\mathbf{p}} \equiv f_{\mathbf{p}} + f_{\mathbf{p}} f^h_{\mathbf{p}} \chi_{\mathbf{p}} \tag{4.47}$$

where f is the equilibrium Fermi distribution and we notice that $f_{\mathbf{p}}(1 - f_{\mathbf{p}}) \propto f'_{\mathbf{p}}$. Linearizing in the small deviation from equilibrium $\chi_{\mathbf{p}}$ one finds upon symmetrization

$$\partial_t \chi_{\mathbf{p}} = -\sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} \mathcal{M}(\mathbf{p}, \mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p} - q) \left(\chi_{\mathbf{p}} + \chi_{\mathbf{p}_1-q} - \chi_{\mathbf{p}_1} - \chi_{\mathbf{p}-q} \right) \tag{4.48}$$

where

$$\begin{aligned}
\mathcal{M}(\mathbf{p}, \mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p} - q) &= \frac{1}{f_{\mathbf{p}} f^h_{\mathbf{p}}} 4V_q[V_q - V_{\mathbf{p}-\mathbf{p}_1}] f_{\mathbf{p}} f_{\mathbf{p}_1-q} f^h_{\mathbf{p}_1} f^h_{\mathbf{p}-q} \times \\
&\quad \times \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q}) \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p} - q])
\end{aligned} \tag{4.49}$$

and we used that for the equilibrium distribution functions $f_{\mathbf{p}} f_{\mathbf{p}_1-q} f^h_{\mathbf{p}_1} f^h_{\mathbf{p}-q} = f^h_{\mathbf{p}} f^h_{\mathbf{p}_1-q} f_{\mathbf{p}_1} f_{\mathbf{p}-q}$ due to energy conservation.

4.4.2 Linearized interworld kinetic equation

Proceeding in a similar way for the interworld collision integral one arrives at

$$\begin{aligned}\mathcal{I}[F_p^{ud}] = \sum_{\mathbf{q},\mathbf{p}_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}](F^{ud}_{\mathbf{p}-\mathbf{q}} F^{ud}_{\mathbf{p}_1} F^{du}_{\mathbf{p}_1-\mathbf{q}} - \\ - \{[F_{\mathbf{p}_1} F_{\mathbf{p}_1-\mathbf{q}} - 1] + [F_{\mathbf{p}_1-\mathbf{q}} - F_{\mathbf{p}_1}] F_{\mathbf{p}-\mathbf{q}}\} \, F^{ud}_{\mathbf{p}}) \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q}) \times \\ \times \, \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p} - q]), \end{aligned} \tag{4.50}$$

and analogously for $\mathcal{I}[F_p^{du}]$.

$$\begin{aligned}\mathcal{I}[F_p^{du}] = \sum_{\mathbf{q},\mathbf{p}_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}](F^{du}_{\mathbf{p}-\mathbf{q}} F^{du}_{\mathbf{p}_1} F^{ud}_{\mathbf{p}_1-\mathbf{q}} - \\ - \{[F_{\mathbf{p}_1} F_{\mathbf{p}_1-\mathbf{q}} - 1] + [F_{\mathbf{p}_1-\mathbf{q}} - F_{\mathbf{p}_1}] F_{\mathbf{p}-\mathbf{q}}\} \, F^{du}_{\mathbf{p}}) \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q}) \times \\ \times \, \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p} - q]). \end{aligned} \tag{4.51}$$

Linearization around the trivial solution $F_p^{du} = \chi^{du}_{\mathbf{p}}$ and $F_p^{ud} = \chi^{ud}_{\mathbf{p}}$ gives

$$\partial_t \chi_{\mathbf{p}} = -\gamma_{\mathbf{p}} \chi_{\mathbf{p}} \mathbf{q} \tag{4.52}$$

where

$$\begin{aligned}\gamma_{\mathbf{p}} =& 4 \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} V_q[V_q - V_{\mathbf{p}-\mathbf{p}_1}] f_{\mathbf{p}_1-q} f^h_{\mathbf{p}_1} f^h_{\mathbf{p}-q} \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q}) \times \\ & \times \, \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p} - q]) \end{aligned} \tag{4.53}$$

is the out-scattering rate.

To linearize around the non-trivial solution, we parametrize $F^{ud} = F^{ud}_{eq} + \delta F^{ud}$ and $F^{du} = F^{du}_{eq} + \delta F^{du}$ where $\delta F^{du}_p = (F_{\mathbf{p}} - 1)\chi^{du}_{\mathbf{p}}$ and $\delta F^{ud}_p = (F_{\mathbf{p}} + 1)\chi^{ud}_{\mathbf{p}}$. We use here the relation between the bosonic and fermionic distribution functions already used in previous calculations in order to rewrite the term with $F^{ud}_{\mathbf{p}}$ and $F^{du}_{\mathbf{p}}$ without the $\mathbf{p}$ dependence, so that we can get a single term of the distribution functions and obtain

$$
\begin{aligned}
\mathcal{I}_{F^{ud}} = & \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} V_q[V_q - V_{\mathbf{p}-\mathbf{p}_1}](F_{\mathbf{p}-q}+1)(F_{\mathbf{p}_1}+1)(F_{\mathbf{p}_1-q}-1)\times \\
& \times \left(\chi^{ud}_{\mathbf{p}-q} + \chi^{ud}_{\mathbf{p}_1} + \chi^{du}_{\mathbf{p}_1-q} - \chi^{ud}_{\mathbf{p}}\right) \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q})\times \\
& \times \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p}-q]), \\
= & -8 \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} V_q[V_q - V_{\mathbf{p}-\mathbf{p}_1}] f_{\mathbf{p}_1-q} f^h_{\mathbf{p}_1} f^h_{\mathbf{p}_1}\times \\
& \times \left(\chi^{ud}_{\mathbf{p}-q} + \chi^{ud}_{\mathbf{p}_1} + \chi^{du}_{\mathbf{p}_1-q} - \chi^{ud}_{\mathbf{p}}\right) \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q})\times \\
& \times \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p}-q]),
\end{aligned} \tag{4.54}
$$

and similarly

$$
\begin{aligned}
\mathcal{I}_{F^{du}} = & \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} V_q[V_q - V_{\mathbf{p}-\mathbf{p}_1}](F_{\mathbf{p}-q}-1)(F_{\mathbf{p}_1}-1)(F_{\mathbf{p}_1-q}+1)\times \\
& \times \left(\chi^{du}_{\mathbf{p}-q} + \chi^{du}_{\mathbf{p}_1} + \chi^{ud}_{\mathbf{p}_1-q} - \chi^{du}_{\mathbf{p}}\right) \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q})\times \\
& \times \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p}-q]), \\
= & \, 8 \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} V_q[V_q - V_{\mathbf{p}-\mathbf{p}_1}] f_{\mathbf{p}_1-q} f^h_{\mathbf{p}_1} f^h_{\mathbf{p}_1}\times \\
& \times \left(\chi^{du}_{\mathbf{p}-q} + \chi^{du}_{\mathbf{p}_1} + \chi^{ud}_{\mathbf{p}_1-q} - \chi^{du}_{\mathbf{p}}\right) \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_1-q} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}-q})\times \\
& \times \delta(\mathbf{p} + [\mathbf{p}_1 - q] - \mathbf{p}_1 - [\mathbf{p}-q]),
\end{aligned} \tag{4.55}
$$

Taking then that

$$
\partial_t F^{ud}_{\mathbf{p}} = 2 f^h_{\mathbf{p}} \partial_t \chi^{ud}_{\mathbf{p}}, \qquad \partial_t F^{du}_{\mathbf{p}} = -2 f_{\mathbf{p}} \partial_t \chi^{du}_{\mathbf{p}}, \tag{4.56}
$$

this can be written as

$$
\partial_t \chi^{ud}_{\mathbf{p}} = - \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} \mathcal{M}(\mathbf{p}, \mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p} - q) \left(\chi^{ud}_{\mathbf{p}-q} + \chi^{ud}_{\mathbf{p}_1} + \chi^{du}_{\mathbf{p}_1-q} - \chi^{ud}_{\mathbf{p}}\right) \tag{4.57}
$$

$$
\partial_t \chi^{du}_{\mathbf{p}} = - \sum_{\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q} \mathcal{M}(\mathbf{p}, \mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p} - q) \left(\chi^{du}_{\mathbf{p}-q} + \chi^{du}_{\mathbf{p}_1} + \chi^{ud}_{\mathbf{p}_1-q} - \chi^{du}_{\mathbf{p}}\right) \tag{4.58}
$$

where as before

$$\mathcal{M}(\mathbf{p},\mathbf{p}_1-q,\mathbf{p}_1,\mathbf{p}-q)=\frac{1}{f_{\mathbf{p}}f^h_{\mathbf{p}}}4V_q[V_q-V_{\mathbf{p}-\mathbf{p}_1}]f_{\mathbf{p}}f_{\mathbf{p}_1-q}f^h_{\mathbf{p}_1}f^h_{\mathbf{p}_1}\delta(\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{p}_1-q}-\epsilon_{\mathbf{p}_1}-\epsilon_{\mathbf{p}-q})\times \\ \times\,\delta(\mathbf{p}+[\mathbf{p}_1-q]-\mathbf{p}_1-[\mathbf{p}-q]) \tag{4.59}$$

and we used again that for the equilibrium distribution functions $f_{\mathbf{p}}f_{\mathbf{p}_1-q}f^h_{\mathbf{p}_1}f^h_{\mathbf{p}-q} = f^h_{\mathbf{p}}f^h_{\mathbf{p}_1-q}f_{\mathbf{p}_1}f_{\mathbf{p}-q}$

4.5 Instability and Lyapunov rate

The main goal of the previous calculations was to derive the linearized intraworld kinetic equations. This now allows us to quantify the quantum butterfly effect by studying eigenvalues of the linearized collision integral. The mains goal is to show that the perturbation of the non-trivial equilibrium solution in the interworld case exponentially grows in time. The growth rate is set by the eigenvalues of the linearized collision integral and defines the Lyapunov rates.
In order to do this calculation we will follow two references that have already made a similar calculation for the linearized intraworld kinetic equations. Only in their case the spectrum of the linearized collision integral is negative and perturbations from equilibrium solutions exponentially relax to zero. The two references we will follow here are the work by Jensen, Smith and Wilkins [12] which also mentioned the work of Abrikosov and Khalatnikov [13].

The collision integral of equations (4.57) and (4.58) can be written in a way that we have the sums substituted by integrals over momenta like $\sum = \frac{1}{(2\pi)^d}\int d\tau^d_p$. Since they both have the same structure I will drop for now the indices for "ud" and "du" for simplicity of calculation, and so we have:

$$\partial_t\chi_{\mathbf{p}} = -\frac{1}{f_{\mathbf{p}}f^h_{\mathbf{p}}}\int d\tau_{\mathbf{p}_1-q}\int d\tau_{\mathbf{p}_1}\int d\tau_{\mathbf{p}-q}4V_q[V_q-V_{\mathbf{p}-\mathbf{p}_1}]f_{\mathbf{p}}f_{\mathbf{p}_1-q}f^h_{\mathbf{p}_1}f^h_{\mathbf{p}_1}\times \\ \times\left(\chi_{\mathbf{p}-q}+\chi_{\mathbf{p}_1}+\chi_{\mathbf{p}_1-q}-\chi_{\mathbf{p}}\right)\delta(\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{p}_1-q}-\epsilon_{\mathbf{p}_1}-\epsilon_{\mathbf{p}-q})\delta(\mathbf{p}+[\mathbf{p}_1-q]-\mathbf{p}_1-[\mathbf{p}-q]) \tag{4.60}$$

Considering a three-dimensional system we have here 9 variables to be integrated, but conservation of the total energy and momentum allows to reduce this number to five. With this we can eliminated one of the $d\tau$ to be integrated. In this case it is convenient to realize the $d\tau_{\mathbf{p}-q}$ for the delta function. We are then left with two remaining three-dimensional integrals. Now, as explained in [13], we can considered two planes in momentum space spanned by the incomming and outgoing momentum vectors participating in the collision. These planes are rotated by an angle ϕ. With this in mind, we notice that the angle between $\mathbf{p}$ and $\mathbf{p}_1 - q$ is approximatedly the same as that between $\mathbf{p}_1$ and $\mathbf{p} - q$, and will be called θ in the following. Considering then also that all momentum states are located in proximity to the Fermi surface, $d\tau_{\mathbf{p}_1} = (m*/p_F)d\epsilon_{\mathbf{p}_1}$ and $d\tau_{\mathbf{p}-q} = (m*/p_F)d\epsilon_{\mathbf{p}-q}$, and we have

$$d\tau_{\mathbf{p}_1} = \frac{1}{(2\pi)^3 p_F}\frac{m*}{cos\frac{1}{2}\theta}(\kappa T)^2 dxdyd\phi, \tag{4.61}$$

where p_F is the momentum at the Fermi surface, $x = \frac{\epsilon_{\mathbf{p}_1} - \mu}{\kappa T}$, $y = \frac{\epsilon_{\mathbf{p}-q} - \mu}{\kappa T}$ and m* is the effective mass of fermions taken from the relation between the Fermi surface momentum and velocity. Here $\frac{\theta}{2}$ comes from the transformation of the integration to a system of cylindrical coordinates, where the main axis is given by the direction of $\mathbf{p} + \mathbf{p}_1 - q$, the radius by the Fermi momentum times $sin\frac{\theta}{2}$, and $\frac{\theta}{2}$ is the angle between all of the momenta with $\mathbf{p} + \mathbf{p}_1 - q$ (to be justified in a moment). The integration element then can be transformed from the cylindrical coordinates to one depending on $\mathbf{p}_1$ and $\mathbf{p} - q$. Using elementary geometrical relations tells us that $df_z \cos\frac{\theta}{2} \approx d\tau_{\mathbf{p}_1}$ and $df_\rho \sin\frac{\theta}{2} \approx d\tau_{\mathbf{p}-q}$, see also Ref. [13]. Now, the approximation that the angle $\frac{\theta}{2}$ is the angle between the momenta with the $\mathbf{p} + \mathbf{p}_1 - q$ axis comes from the fact that the momenta are all near the Fermi surface. That is, in order to have the relation that states $\mathbf{p} + \mathbf{p}_1 - q = \mathbf{p}_1 + \mathbf{p} - q$ we need then to have this angle with respect to the $\mathbf{p} + \mathbf{p}_1 - q$ direction. If one of the momenta on the left hand side would have a angle significantly different from the other, with respect to the $\mathbf{p} + \mathbf{p}_1 - q$ direction, it would have a bigger magnitude and therefore lie out of the Fermi surface, inorder to compensate for the sum relation. Finally, to deal with the remaining integrals we use spherical coordinates, which gives $d\tau_{\mathbf{p}_1 - q} = (\mathbf{p}_1 - q)^2 d(\mathbf{p}_1 - q)\sin\theta d\theta d\phi_2 = p_F m * \kappa T d\sin\theta d\theta d\phi_2$, with $z = \frac{(\epsilon_{\mathbf{p}_1 - q} - \mu)}{\kappa T}$. Our kinetic equation can be written as

$$\frac{1}{4\cosh^2(\frac{1}{2}\bar{t})}\frac{\partial_t\chi_{\mathbf{p}}}{\kappa T} = \frac{1}{(2\pi)^6}m*^3(\kappa T)\int\int\int dxdydz\int_0^\pi \sin\frac{1}{2}\theta d\theta\int_0^\pi d\phi\int_0^{2\pi}d\phi_2\times$$
$$\times V_q(V_q - V_{\mathbf{p}-\mathbf{p}_1})\delta(t+z-x-y)f_{\mathbf{p}}f_{\mathbf{p}_1-q}f^h_{\mathbf{p}_1}f^h_{\mathbf{p}_1}[\chi_{\mathbf{p}} - \chi_{\mathbf{p}_1-q} - \chi_{\mathbf{p}_1} - \chi_{\mathbf{p}-q}] \tag{4.62}$$

where $\bar{t} = \frac{(\epsilon_{\mathbf{p}}-\mu)}{\kappa T}$, $1/\tau_0 = \frac{m*^3(\kappa T)^2}{8\pi^4}V_q(V_q - V_{\mathbf{p}-\mathbf{p}_1})$ and the hyperbolical function is another way of writing the product of particle and hole distributions.
We now look for an ansatz for χ. A convenient one is $\chi_{\mathbf{p},\bar{t}}(t) = 2\tau_0\cos(\frac{1}{2}\bar{t})Q_t(\bar{t})$, that goes for any other variable x, y and z. We are now interested in the dependence on t, that is the temporal evolution Q_t of the perturbation. The expression simplify noting that the collision integral remains the same under the exchange of x and y and x and $-z$, and noting that (for any fixed t) Q_t is an odd function. Using further the following relations

$$\int\int\int dxdydz\delta(t+z-x-y)f(\bar{t})f_z f^h(x)f^h(y) = \frac{1}{2}f(\bar{t})f^h(\bar{t}))(\pi^2+\bar{t}^2)$$

$$\int\int dydz\delta(t+z-x-y)f(\bar{t})f_z f^h(x)f^h(y) = \frac{1}{2}f(\bar{t})f^h(\bar{t}))\frac{\cosh\frac{1}{2}\bar{t}}{\cosh\frac{1}{2}x}\frac{x-\bar{t}}{2\sinh\frac{1}{2}(x-\bar{t})}$$

we then get the kinetic equation as:

$$2\tau_0\partial_t Q_t(\bar{t}) = (\pi^2+\bar{t}^2)Q_t(\bar{t}) - \int dxQ_t(x)F(\bar{t}-x), \qquad F(\bar{t}-x) = \frac{(\bar{t}-x)}{2\sinh\frac{1}{2}(\bar{t}-x)} \tag{4.63}$$

We can now read the right hand side as a linear equation for eigenvectors which we wish to diagonalize. To this end we Fourier transform the right hand side in $\bar{t}$. The factor $\bar{t}^2$ then becomes a second derivative in the conjugate variable that we will call q and the convolution integral becomes the product with an inverse hyperbolic function. We then want to find the eigensolutions of this combined operator. That is we arrive at the eigenequation

$$\left(-\frac{d^2}{dq^2} + 1 - \frac{\lambda_n}{\cosh^2 q}\right)\bar{\phi}_n = E_n\bar{\phi}_n. \tag{4.64}$$

with λ_n as $\lambda_n = n(n+1)$. We recognize here a Schroedinger equation with a hyperbolical potential, and the corresponding energies will set the Lyapunov rates. That is the spectrum of this operator defines how perturbation Q_t will increase (decrease) in time for

$E_n > 0$ ($E_n < 0$). Reminding that solutions should fulfill the normalization condition

$$\int d\bar{t}\phi_n(\bar{t})\phi_m(\bar{t}) = \delta_{n,m}$$

we can decompose any perturbation Q_t in terms of the corresponding eigenfunctions. It is known that solutions of the above equation are given by associated Legendre polynomials P_n^1,

$$\left(-\frac{d^2}{dq^2} + 1 - \frac{\lambda_n}{\cosh^2 q}\right) c_n P_n^1(\tanh q) = E_n c_n P_n^1(\tanh q) \tag{4.65}$$

Substituting e.g. $\tanh q$ in $P_2^1(x) = (-3x\sqrt{1-x^2})$ we get

$$\left(-\frac{d^2}{dq^2} + 1 - \frac{\lambda_n}{\cosh^2 q}\right) c_2(-3\tanh q sechq) = [1 - sech^2 q]c_2(-3\tanh q sechq) \tag{4.66}$$

and Fourier transforming then back to the original coordinates we have

$$\partial_t(2\tau_0 Q_t(\bar{t})) = \left(1 - \frac{\bar{t}}{\sinh \bar{t}}\right) Q_t(\bar{t}) \tag{4.67}$$

That is, the time dependence t of the perturbation

$$Q_t(\bar{t}) \approx \exp(\lambda_{\bar{t}}\, t) \tag{4.68}$$

where

$$\lambda_{\bar{t}} = \frac{1}{2\tau_0}\left(1 - \frac{\bar{t}}{\sinh \bar{t}}\right) \tag{4.69}$$

is the rate of exponential growth of the perturbation. A similar calculation holds for other eigenfunctions of the Schroedinger problem (4.64). That is, we here see the manifestation of many-body chaos in the instability of the non-trivial intraworld solution which explodes on a rate set by $1/\tau_0$.

Discussion and Outlook

In this thesis we have studied signatures of chaos in many-body quantum systems. Using a generalized Keldysh formalism we have derived generalized kinetic equations which describe distribution functions that store information on out-of-time ordered correlation functions. The latter have recently moved into the focus of studies of many-body chaos, since in the presence of many-body chaos they show an exponential growth which can be viewed as a quantum version of the butterfly effect. Within the present approach the quantum butterfly effect manifests in an instability of equilibrium solutions to the generalized kinetic equations. That is, perturbation around equilibrium solutions exponentially blow up in time. This then allows for a systematic study of Lyapunov rates from the spectrum of the linearized collision integral.

We have applied the machinery to the weakly interacting electron gas in three dimensions. In this case we verified the exponential increase of small perturbations to equilibrium solutions which are set by the scale $1/\tau_0 \propto T^2$. One can verify that in case of a contact interaction and spin-polarized electrons the proportionality constant vanishes due to a cancellation of direct and exchange terms. This is to be expected since the Pauli principle in this case forbids any interactions and thus suppresses many-body chaos. For general cases the Lyapunov rate increases quadratically in temperature, as was seen in its calculations according to its dependence [13]. This is similar as in the inelastic scattering rate and can be traced back to the phase space volume available for two-particle scattering processes.

The work done here is the first step and will serve as the basis for future investigation. Our interest is to study situations under which one may expect a suppression of many-body chaos, maybe in disordered systems. As discussed in several recent works, even in the presence of (weak) interactions electrons in a disordered system may become localized. That is, the disordered interacting electron gas can make a phase transition into a non ergodic “many-body" localized phase. It would be very interesting to see wether signatures of a suppression of many-body chaos can be observed within the present formalism. In the present formalism this should be observed in the spectrum of the linearized collision integral.

A second kind of system that we want to study is one-dimensional system. It is known that for a linearized spectrum two-particle scattering process cannot relax the system, and therefore no many-body chaos can be expected. One can take this as a starting point and include effects of curvature and three particle collision which should then serve as a trigger for many-body chaos [17]. Again it would be interesting to see wether one can investigate the transition from the non-ergodic to the many-body chaotic phase as a function of curvature and temperature from the spectrum of the linearized collision integral.

Appendix A

Discussing the origin of the instability

Now it would be my interest to calculate the sum and subtraction between the contributions for the "up to down" and "down to up" worlds. Using the equations (4.57) and (4.58) we should get

$$\partial_t \left(\chi^{ud}_{\mathbf{p}} - \chi^{du}_{\mathbf{p}}\right) = -\sum_{\mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p}-q} \mathcal{M}(\mathbf{p}, \mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p} - q)[(\chi^{ud}_{\mathbf{p}-q} - \chi^{du}_{\mathbf{p}-q}) + (\chi^{ud}_{\mathbf{p}_1} - \chi^{du}_{\mathbf{p}_1}) - \\ - (\chi^{ud}_{\mathbf{p}_1 - q} - \chi^{du}_{\mathbf{p}_1 - q}) - (\chi^{ud}_{\mathbf{p}} - \chi^{du}_{\mathbf{p}})] \tag{A.1}$$

$$\partial_t \left(\chi^{ud}_{\mathbf{p}} + \chi^{du}_{\mathbf{p}}\right) = -\sum_{\mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p}-q} \mathcal{M}(\mathbf{p}, \mathbf{p}_1 - q, \mathbf{p}_1, \mathbf{p} - q)[(\chi^{ud}_{\mathbf{p}-q} + \chi^{du}_{\mathbf{p}-q}) + (\chi^{ud}_{\mathbf{p}_1} + \chi^{du}_{\mathbf{p}_1}) - \\ + (\chi^{ud}_{\mathbf{p}_1 - q} + \chi^{du}_{\mathbf{p}_1 - q}) - (\chi^{ud}_{\mathbf{p}} + \chi^{du}_{\mathbf{p}})]. \tag{A.2}$$

Which are the subtraction and sum of the mentioned equations, respectively. Now, it is possible to see that while the first equation reproduces the structure of the "intraworld" kinetic equation, although with opposite sign, the second one takes the form of the "interworld" kinetic equations. This is interesting because now we have the possibility of use of just one equivalent equation for the interworld terms. Lets now recover the linearization for the interworld kinetic equations, given by $F^{ud} = F^{ud}_{eq} + \delta F^{ud}$ and $F^{du} = F^{du}_{eq} + \delta F^{du}$ where $\delta F^{du}_p = (F_{\mathbf{p}} - 1)\chi^{du}_{\mathbf{p}}$ and $\delta F^{ud}_p = (F_{\mathbf{p}} + 1)\chi^{ud}_{\mathbf{p}}$ or more generally: $F^{ud} = F^{ud}_{eq} + F^{ud}\chi$ and $F^{du} = F^{du}_{eq} + F^{du}\chi$, and with this remember that, since $F_p = 1 - 2n_p$ then we have

$F^{ud} = 2\tilde{n}_p$ and $F^{du} = -2n_p$. Notice that when we do the calculations for the linearization of the interworld terms what happens is that we directly substitute the term of "up" and "down" for the expressions for the Fermi distribution functions, then arriving at the three product for holes and particles. The left side (or "the pure interworld side") naturally becomes the product while the right side need the use of the bosonic relation to eliminate the "p" dependence, but just in equilibrium. It is the following:

$$\mathcal{I}[F^{ud}_{eq} + F^{ud}\chi] = \sum_{\mathbf{q},\mathbf{p}_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}]((F^{ud}_{eq} + F^{ud}\chi)_{\mathbf{p}-\mathbf{q}}(F^{ud}_{eq} + F^{ud}\chi)_{\mathbf{p}_1}(F^{du}_{eq} + F^{du}\chi)_{\mathbf{p}_1-\mathbf{q}} \\ - \{[F_{\mathbf{p}_1}F_{\mathbf{p}_1-\mathbf{q}} - 1] + [F_{\mathbf{p}_1-\mathbf{q}} - F_{\mathbf{p}_1}]F_{\mathbf{p}-\mathbf{q}}\}(F^{ud}_{eq} + F^{ud}\chi)_{\mathbf{p}})\delta_{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{p}_1-\mathbf{q}},\epsilon_{\mathbf{p}_1}+\epsilon_{\mathbf{p}-\mathbf{q}}}, \tag{A.3}$$

and similar for $\mathcal{I}[F^{du}_p]$

$$\mathcal{I}[F^{du}_{eq} + F^{du}\chi] = \sum_{\mathbf{q},\mathbf{p}_1} V_{\mathbf{q}}[V_{\mathbf{q}} - V_{\mathbf{p}-\mathbf{p}_1}]((F^{du}_{eq} + F^{du}\chi)_{\mathbf{p}-\mathbf{q}}(F^{du}_{eq} + F^{du}\chi)_{\mathbf{p}_1}(F^{ud}_{eq} + F^{ud}\chi)_{\mathbf{p}_1-\mathbf{q}} \\ - \{[F_{\mathbf{p}_1}F_{\mathbf{p}_1-\mathbf{q}} - 1] + [F_{\mathbf{p}_1-\mathbf{q}} - F_{\mathbf{p}_1}]F_{\mathbf{p}-\mathbf{q}}\}(F^{du}_{eq} + F^{du}\chi)_{\mathbf{p}})\delta_{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{p}_1-\mathbf{q}},\epsilon_{\mathbf{p}_1}+\epsilon_{\mathbf{p}-\mathbf{q}}}. \tag{A.4}$$

It is possible to notice that the pure equilibrium parts of this collision integral will vanish, if we use on the right side the bosonic relation to eliminate the "p" dependence. Then it will leave the collision integral as a equation for the first order deviation terms where the coefficients are equal for each of the four χ and given by the product of three Fermi distributions.But before we substitute this common term by the Fermi distributions we could have written the terms as $(F_{\mathbf{p}-q}+1)(F_{\mathbf{p}_1}+1)(F_{\mathbf{p}_1-q}-1)$ and $(F_{\mathbf{p}-q}-1)(F_{\mathbf{p}_1}-1)(F_{\mathbf{p}_1-q}+1)$ for each world. The expansion of these two terms give as a result, respectively:

$$ud : F_{\mathbf{p}-q}F_{\mathbf{p}_1}F_{\mathbf{p}_1-q} + F_{\mathbf{p}_1-q} - F_{\mathbf{p}-q} - F_{\mathbf{p}_1} + (F_{\mathbf{p}_1}F_{\mathbf{p}_1-q} + F_{\mathbf{p}-q}F_{\mathbf{p}_1-q} - F_{\mathbf{p}-q}F_{\mathbf{p}_1} - 1) \tag{A.5}$$

$$du : F_{\mathbf{p}-q}F_{\mathbf{p}_1}F_{\mathbf{p}_1-q} + F_{\mathbf{p}_1-q} - F_{\mathbf{p}-q} - F_{\mathbf{p}_1} - (F_{\mathbf{p}_1}F_{\mathbf{p}_1-q} + F_{\mathbf{p}-q}F_{\mathbf{p}_1-q} - F_{\mathbf{p}-q}F_{\mathbf{p}_1} - 1) \tag{A.6}$$

These two equations are written in a convenient way so that we have a equal part and a equal but with opposite signal part. The equal one it is actually the Keldysh term of the self-energy of the Keldysh component, Σ^K, as in equation (4.20). The other part is equal to the self-energy difference between the Retarded and Advanced ones, as in equation (4.21).In the intraworld case these parentheses terms are splitted, part being integrated in the called "in" term of the collision integral and the other the "out" term which are symmetrical in principle, that is, equal contributions mostly if you are analysing in equilibrium. But in the interworld case these terms vanish in the equilibrium parts, but not in the linearized one. Since the Fermi distributions are factorized from each χ contribution, in a way that is not separated as an "in" and "out" but as a full factor in the coefficient of the collision integral, being $F_{\mathbf{p}_1}F_{\mathbf{p}_1-q} + F_{\mathbf{p}-q}F_{\mathbf{p}_1-q} - F_{\mathbf{p}-q}F_{\mathbf{p}_1} - 1$. Or in a more interesting way to write: $(n_{\mathbf{p}_1}n_{\mathbf{p}_1-q} + n_{\mathbf{p}-q}n_{\mathbf{p}_1-q} - n_{\mathbf{p}_1}n_{\mathbf{p}-q}) - n_{\mathbf{p}_1-q}$.
Linearizing in the same way we linearized the kinetic equation for the intraworld case we get that

$$f_{\mathbf{p}-q}(f_{\mathbf{p}_1-q} - f_{\mathbf{p}-q})\chi_{\mathbf{p}-q} + f_{\mathbf{p}_1}(f_{\mathbf{p}_1-q} - f_{\mathbf{p}-q})\chi_{\mathbf{p}_1} + f_{\mathbf{p}_1-q}(f_{\mathbf{p}-q} + f_{\mathbf{p}_1} + 1)\chi_{\mathbf{p}_1-q} \tag{A.7}$$

But this structure is achieved by considering that $(f_{\mathbf{p}_1}f_{\mathbf{p}_1-q} + f_{\mathbf{p}-q}f_{\mathbf{p}_1-q} - f_{\mathbf{p}_1}f_{\mathbf{p}-q}) - f_{\mathbf{p}_1-q}$ goes to zero in equilibrium, which does not occur directly, so we shall impose this to be zero. Doing this we find relations between the distribution function that we should substitute in the equation (A.8) for each deviation, since they are independent of each other, then we get

$$(f_{\mathbf{p}_1-q}f^h_{\mathbf{p}_1})\chi_{\mathbf{p}-q} + (f_{\mathbf{p}_1-q}f^h_{\mathbf{p}-q})\chi_{\mathbf{p}_1} + (f_{\mathbf{p}-q}f_{\mathbf{p}_1})\chi_{\mathbf{p}_1-q} \tag{A.8}$$

Now it would be of our interest to now what is the origin the instability of the system considering the two worlds to be considered. We can see this by analysing, for example, the direct diagram Figure (4.2).
Here you can see that the deviations (A.8) can be analysed with the use of the diagrams, in a way that the first deviation, the $\chi_{\mathbf{p}-q}$ has on the diagram one the firs component as orientated from up to down, and the second one from down to up. The second one, $\chi_{\mathbf{p}_1}$ has the same up to down from the previous and the second is the $\mathbf{p} - q$ from down to up. In both cases mutual "conversation" between the world. But in the last one,

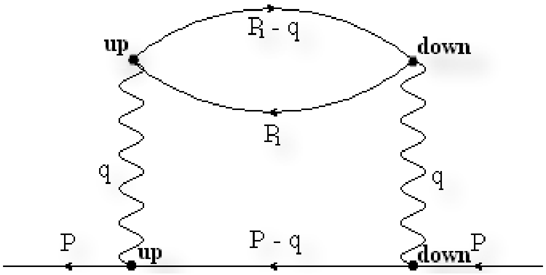

Figure A.1: Note here that in accordance with the previously developed theory "a" and "b" has to be within the same world as "c" and "d". That is, $c = d$ both “up" or both “down", and $a = b$ down or up, for the down-up case, or vice-versa, respectively.

$\chi_{\mathbf{p_1}-q}$, we have both contributions orientated from down to up. A unbalance in these contributions. Now notice that we would expect for the other case the vertically inverted symmetrically diagram in the "ups" and "downs", and so we would the same type of contributions orientated from up to down. In the equations this meant a signal change as saw in equations (A.6).

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